

ON THE FIRST ORDER ASYMPTOTIC THEORY OF QUANTUM ESTIMATION

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ABSTRACT

We give a rigorous treatment on the foundation of the first order asymptotic theory of quantum estimation, with tractable and reasonable regularity conditions. Different from past works, we do not use Fisher information nor MLE, and an optimal estimator is constructed based on locally unbiased estimators. Also, we treat state estimation by local operations and classical communications (LOCC), and estimation of quantum operations.

1. INTRODUCTION

The purpose of this paper is to give a rigorous foundation of the first order asymptotic theory of quantum estimation, which has been established in these years. In addition to most basic setting, we also treat state estimation by local operations and classical communications (LOCC, in short) and estimation of quantum operations.

This research field was initiated by Nagaoka(1987), Nagaoka(1989), followed by Hayashi and Matsumoto(1998), Gill and Massar(2002). (Many of important papers in the field are included in Hayashi(2005).) Relying on classical estimation theory, especially the fact that the inverse of Fisher information gives the optimal efficiency of consistent estimators, they had reduced the optimization of consistent estimators to optimization of Fisher information, or equivalently, of locally unbiased estimators. These works had laid foundation on which number of works, mostly computation of asymptotically optimal estimators and their costs, are based. In closer look, however, they either miss the detail of the proof, or assume intractable regularity conditions.

One reason for such incompleteness is that the focus of these works were consequences of the foundations, rather than their rigorous proof. Also, the following technical difficulties seems to be a part of reasons. In quantum statistics, the probability distribution of the data depends on the choice of measurement. Therefore, for classical estimation theory to be applicable, a set of regularity conditions should hold for *all* the probability distributions resulting from arbitrary measurement of interest. In Hayashi and Matsumoto(1998), they use this sort of statement as their regularity condition. As a result, their regularity conditions are quite difficult

to check for given quantum statistical models.

The purpose of the paper is to provide rigorous proof assuming tractable regularity conditions, including the case of infinite dimensional Hilbert space. In addition to the most basic settings, we also treat state estimation by semi-classical measurement and by local operations and classical communications (LOCC, in short). Also, estimation of quantum operations is studied.

Different from previous works, we avoided use of Fisher information, and composed an asymptotically efficient estimator from an optimal locally unbiased estimator, because of the following reasons. First, quantum asymptotic Crammer-Rao bound is not a simple function of any quantum analogue of Fisher information. It equals Holevo bound, which is defined in terms of operator version of asymptotically unbiasedness conditions (Hayashi and Matsumoto (2004), Matsumoto (1999), Guta and Jencova (2006)). The second motivation is to simplify the regularity conditions, by avoiding technical difficulties stated above.

One of major difference between quantum mechanics and classical mechanics is behavior of composite systems. In quantum mechanics, the state of the system and the measurement in composite systems may not be in convex combinations of those without correlations between subsystems. In such cases, we often observe non-trivial quantum effects, which can *never* be reproduced by classical mechanical random variables, such as violation of Bell's inequality. Therefore, it is of interest to compare measurement with non-trivial correlations and the one without it in their efficiency of state estimation.

For that purpose, we study *semi-classical* measurements and *LOCC* (, short for local operations and classical communications,) measurements. In the former, we are not allowed to use measurement collectively acts on given n independent samples. In the latter, each sample is a state in a composite system (A and B, say), and we are not allowed to use the measurement quantumly correlating over A-B split.

The last topic is estimation of a quantum operations. It had been observed that for some cases (e.g., unitary operations, or noiseless operations), the mean square error of optimal estimators scales as $O(1/n^2)$ (Heisenberg rate), which is significantly smaller than $O(1/n)$, and there had been suggestion of efficient measurement scheme utilizing this effect. Recently, however, several authors (Fujiwara (2005), Zhengfeng Ji, et. al. (2006), etc) had pointed out that $O(1/n^2)$ -scaling is not observed in some class of operations (typically, they corresponds to noisy operations). We show that $O(1/n^2)$ -scaling is rather exceptional, and not observed so long as the model lies in interior of the totality of quantum operations.

2. QUANTUM ESTIMATION THEORY

2.1. QUANTUM STATE AND MEASUREMENT

In quantum mechanics, the probability distribution of data $z \in \mathbb{R}^l$ is a function of the *state* ρ of the system of interest, and the *measurement* M which is applied to the system. The probability that ω lies in a Borel set Δ , the corresponding random variable, and the post-measurement state is denoted by $P_\rho^M(\Delta)$, Ω , and ρ_Δ^M , respectively. (Throughout the paper, the random variable is denoted by capital letters, and the elements of

its range is denoted by its decapitalization.)

ρ and \mathbf{M} are represented by linear operators defined in a separable Hilbert space (\mathcal{H} , say). The inner product of φ and ψ is denoted by $\langle \varphi, \psi \rangle$. We assign to the composite of the system \mathcal{H}_1 and \mathcal{H}_2 the tensor product $\mathcal{H}_1 \otimes \mathcal{H}_2$, which is the linear span of $\{e_{1,i} \otimes e_{2,j}\}$ ($\{e_{1,i}\}$ and $\{e_{2,i}\}$ be a complete orthonormal basis (CONS) of \mathcal{H}_1 and \mathcal{H}_2 respectively).

The notation $|A|$ means $|A| := (AA^\dagger)^{1/2}$, and $\|A\|_1 := \text{tr } |A|$ is a quantum version of total variation. The totality of *trace class operators*, or operators with $\|A\|_1 < \infty$, is denoted by $\tau c(\mathcal{H})$. Also, $\|A\| := \sup_{\|\varphi\|=1} \|A\varphi\|$ and $\mathcal{B}(\mathcal{H})$ denotes the totality of *bounded operators*, or operators with $\|A\| < \infty$. (the standard norm in \mathbb{R}^m and in \mathcal{H} is also denoted by $\|\cdot\|$.) We introduce an order in the space of matrices by $A \geq (>) B \Leftrightarrow \langle \varphi, A\varphi \rangle \geq (>) \langle \varphi, B\varphi \rangle, \forall \varphi$. An operator A is said to be *positive*, if $A \geq 0$. A mapping Λ of $\tau c(\mathcal{H})$ to $\tau c(\mathcal{H}')$ is called *completely positive*, if $\Lambda \otimes \mathbf{I} : \mathcal{B}(\mathcal{H} \otimes \mathcal{K}) \rightarrow \mathcal{B}(\mathcal{H}' \otimes \mathcal{K})$ is positive, i.e., $A \geq 0 \Rightarrow \Lambda \otimes \mathbf{I}(A) \geq 0$. Λ is said to be *trace preserving* if $\text{tr } X = \text{tr } \Lambda(X)$ ($\forall X$). Also we define $\|\Lambda\|_{cb} := \sup_{X: \|X\|_1=1} \|\Lambda \otimes \mathbf{I}(X)\|_1$.

A state of the system is represented by a *density operator*, or an operator ρ with $\rho \geq 0$, $\rho = \rho^*$, and $\text{tr } \rho = 1$. A measurement \mathbf{M} is represented by an *instrument*, or a σ -additive map $\mathbf{M} : \Delta \rightarrow \mathcal{M}[\Delta]$ of the collection \mathfrak{B} of Borel subsets in \mathbb{R}^m into a completely positive linear transform $\mathbf{M}[\Delta]$ in $\tau c(\mathcal{H})$ with $\mathbf{M}[\mathbb{R}^l]$'s being trace-preserving. Here, σ -additivity is in the sense of strong operator topology in $\mathcal{B}(\tau c(\mathcal{H}))$. Using ρ and $\mathbf{M}[\Delta]$, $P_\rho^{\mathbf{M}}(\Delta)$ and ρ_Δ is given by $\text{tr } \mathbf{M}[\Delta] \rho$ and $\frac{1}{P_\rho^{\mathbf{M}}(\Delta)} \mathbf{M}[\Delta](\rho)$, respectively. An operation which does not extract information is described by a completely positive and trace-preserving (CPTP) linear map Λ from $\tau c(\mathcal{H})$ to $\tau c(\mathcal{H}')$.

When we are interested only in $P_\rho^{\mathbf{M}}(\Delta)$, we use a *positive operator valued measure* (POVM, in short), or a σ -additive map $M : \Delta \rightarrow \mathcal{M}(\Delta)$ of \mathfrak{B} to positive Hermitian operators with $M(\mathbb{R}^l) = \mathbf{1}$. Here, σ -additivity is in the sense of weak operator topology in $\mathcal{B}(\mathcal{H})$. The POVM M corresponding to the measurement \mathbf{M} satisfy $P_\rho^{\mathbf{M}}(\Delta) = \text{tr } \mathbf{M}[\Delta] \rho = \text{tr } \rho M(\Delta)$. Throughout the paper, POVM of a measurement is denoted by the same character as the measurement but in the standard font.

The *support* $\text{supp}(\mathbf{M})$ of the instrument \mathbf{M} over $\mathfrak{B}(\mathbb{R}^l)$ is the smallest set with $\mathbf{M}[\text{supp}(\mathbf{M})] = \mathbf{M}[\mathbb{R}^l]$. The support of a POVM and a measure over $\mathfrak{B}(\mathbb{R}^l)$ are defined analogously.

In this paper, we need integral of the function taking values in $\tau c(\mathcal{H})$ and $\mathcal{B}(\tau c(\mathcal{H}))$, which is a Banach space with the norm $\|\cdot\|_1$ and $\|\cdot\|_{cb}$, respectively. A Banach space valued function f is called strongly measurable iff $\forall \varepsilon > 0 \exists f' \|f(x) - f'(x)\| < \varepsilon$ holds almost everywhere. f is called weakly measurable iff $\langle y^*, f(x) \rangle$ is measurable for any element y^* of the dual space. Since $\tau c(\mathcal{H})$ and $\mathcal{B}(\tau c(\mathcal{H}))$ are separable, these two concepts are equivalent in our case due to Theorem 1.1.4 of Schwabik and Guoju (2005).

Pettis integral of weakly measurable function f is defined by the relation $\int \langle y^*, f(x) \rangle dx = \langle y^*, \int f(x) dx \rangle, \forall y$. Bochner integral of a simple function $\sum_i c_i \chi_{A_i}$ is defined as $\sum_i c_i \mu(A_i)$. For a strongly measurable

function f , it is defined as $\lim_{n \rightarrow \infty} \int f_n(x) dx$ (convergent in norm), where $\{f_n\}_n$ is a sequence of simple functions with $\lim_{n \rightarrow \infty} \|f_n(x) - f(x)\| = 0$ almost everywhere. Bochner integral exists iff $\int \|f\| dx < \infty$ (Theorem 1.4.3 of Schwabik and Guoju (2005)). Fubini's theorem holds for Pettis integral and Bochner integral.

2.2 ASYMPTOTIC THEORY OF QUANTUM STATE ESTIMATION

Suppose that we are given n independently and identically prepared samples, i.e., the system $\underbrace{\mathcal{H} \otimes \cdots \otimes \mathcal{H}}_n := \mathcal{H}^{\otimes n}$ in the state $\underbrace{\rho_\theta \otimes \cdots \otimes \rho_\theta}_n =: \rho_\theta^{\otimes n}$, where ρ_θ is drawn from a *quantum statistical model* $\mathcal{M} := \{\rho_\theta; \theta \in \Theta\}$, with Θ 's being an open convex region in \mathbb{R}^m .

Our purpose is to estimate the true value of θ , based on a measurement M^n acting in $\mathcal{H}^{\otimes n}$. Based on the measurement result $\omega_n \in \mathbb{R}^{l_n}$, we compute the estimate T_n of θ . The pair $\mathcal{E}_n := \{M^n, T_n\}$ (or sometimes the sequence $\{\mathcal{E}_n\}_{n=1}^\infty$ also) is called an *estimator*. T_n is a measurable function of \mathbb{R}^{l_n} to $\hat{\Theta}_n \subset \mathbb{R}^m$. The following notations are used: $E_\theta^{M^n}[f(\omega_n)] := \int f(\omega_n) \text{tr} \rho_\theta M^n(d\omega_n)$, $(\text{MSE}_\theta[\mathcal{E}_n])_{i,j} := E_\theta^{M^n}(T_n^i - \theta^i)(T_n^j - \theta^j)$, $(V_\theta[\mathcal{E}_n])_{i,j} := E_\theta^{M^n}(T_n^i - E_\theta^{M^n}[T_n^i])(T_n^j - E_\theta^{M^n}[T_n^j])$. Below, Tr denotes the trace over \mathbb{R}^m , and $\partial_j := \frac{\partial}{\partial \theta^j}$. G_θ is a symmetric positive real matrix, and $\theta \rightarrow G_\theta$ is continuously differentiable, $\text{Tr} G_\theta \leq b_1$, and $|\text{Tr} G_\theta - \text{Tr} G_{\theta'}| \leq b_1 \|\theta - \theta'\|$. We also define $(B_{\theta_0}[\mathcal{E}_n])_j^i := \partial_j E_{\theta_0}^{M^n}[T_n^i]|_{\theta=\theta_0}$. Our interest is the first order asymptotic term of the weighted mean square error $\overline{\lim}_{n \rightarrow \infty} n \text{Tr} G_\theta \text{MSE}_\theta[\mathcal{E}_n]$, minimized over *asymptotically unbiased estimator*, or $\{\mathcal{E}_n\}_{n=1}^\infty$ with the following condition:

$$\lim_{n \rightarrow \infty} E_\theta^{M^n}[T_n] = \theta, \quad \lim_{n \rightarrow \infty} (B_\theta[\mathcal{E}_n])_j^i = \delta_j^i, \quad \forall \theta \in \Theta. \quad (1)$$

In considering (1), $E_\theta^{M^n}[T_n]$ has to be differentiable, which is made sure by Lemma 2. Use of MSE may be justified based on the existence of the asymptotic normal efficient estimator, which is composed in Subsection 3.3.

Our purpose is to replace this condition by the following tractable condition without changing the optimal lowerbound to the asymptotic cost: $\mathcal{E}_{\theta_0,n} = \{M_{\theta_0,n}^n, T_{\theta_0,n}\}$ is said to be *locally unbiased at θ_0* if

$$E_{\theta_0}^{M_{\theta_0,n}^n}[T_{\theta_0,n}] = \theta_0, \quad (B_{\theta_0}[\mathcal{E}_{\theta_0,n}])_j^i = \delta_j^i. \quad (2)$$

Note that the condition (2) is closed at the point θ_0 . In the following sections, we prove that minimization of $\overline{\lim}_{n \rightarrow \infty} n \text{Tr} G_\theta \text{MSE}_\theta[\mathcal{E}_n]$ over all the asymptotically unbiased estimators can be reduce to minimization over the locally unbiased estimators under some proper regularity conditions.

3. THE BASIC SETTING

3.1. REGULARITY CONDITIONS AND ASYMPTOTIC CRAMER-RAO BOUND

Regularity conditions on quantum statistical models and estimators are listed in Table 1, in which convergence is with respect to $\|\cdot\|_1$. $\Diamond_{i,\theta,n}$ is as defined in Lemma 3, and $\Diamond_{i,\theta,n}^{(1)} := \Diamond_{i,\theta,n} \otimes \rho_\theta^{\otimes n-1} + \rho_\theta \otimes \Diamond_{i,\theta,n} \otimes \rho_\theta^{\otimes n-2} + \cdots$.

(M.1)	$\partial_i \rho_\theta$ and $\partial_i \partial_j \rho_\theta$ exist and are locally uniformly continuous. $\ \partial_i \rho_{\theta_0}\ , \ \partial_i \partial_j \rho_\theta\ _1 \leq a_1 < \infty$.
(M.2)	$\exists L_{\theta,i}$: Hermitian and $\partial_i \rho_\theta = \frac{1}{2} (L_{\theta,i} \rho_\theta + \rho_\theta L_{\theta,i})$, and $\text{tr} \rho_\theta (L_{\theta,i})^2 < \infty, \forall \theta \in \Theta$.
(M.3)	There is an estimator $\tilde{\mathcal{E}}_n = \{\tilde{\mathbf{M}}^n, \tilde{T}_n\}$ in $\mathcal{H}^{\otimes n}$, such that
(M.3.1)	(1) and (E) are satisfied.
(M.3.2)	$\mathbb{E}_{\tilde{\mathbf{M}}^n} \left\ \tilde{T}_n - \theta \right\ ^4 \leq \frac{D_{\theta,2}}{n^2}, \forall \theta \in \Theta, \exists D_{\theta,2}$.
(M.3.3)	$\tilde{\mathbf{M}}^n$ is n times repetition of a measurement $\tilde{\mathbf{M}}$ in \mathcal{H} , producing the data $x_\kappa \in \mathbb{R}^l$.
(E)	$\exists a_{4,n}, \forall \theta \in \Theta, \int \ T_n(\omega_n) - \theta\ \text{tr} \diamond_{\theta,n}^{(1)} M^n(d\omega_n) \leq n a_1 a_{4,n}, \int \ T_n(\omega_n) - \theta\ ^2 \text{tr} \rho_\theta^{\otimes n} M^n(d\omega_n) \leq n a_{4,n}^2$.
(E')	T_n takes values in $\hat{\Theta}_{T_n}$, with $\sup_{\theta, \theta' \in \hat{\Theta}_{T_n}} \ \theta - \theta'\ \leq a_{4,n} < \infty$.

Table 1: Regularity conditions on quantum statistical models (M.1-4) and estimators (E), (E')

Among the conditions on models, only (M.1) is needed to prove the lowerbound. *Unless otherwise mentioned, (M.1) are assumed throughout the paper.* (M.2-3) are necessarily to prove the achievability of the lowerbound. (M.2) is equivalent to $|\partial_i \text{tr} \rho_\theta X| \leq c |\text{tr} \rho_\theta X^2|$ for any bounded Hermitian.

If $\dim \mathcal{H} < \infty$, an example of estimator $\tilde{\mathcal{E}}_n = \{\tilde{\mathbf{M}}^n, \tilde{T}_n\}$ with (M.3.1-3) is constructed as follows. Let $l := (\dim \mathcal{H})^2$, and define $\mathbf{e}_v := (0, \dots, 0, \overset{v}{1}, 0, \dots, 0) \in \mathbb{R}^l$. Let $\text{supp}(\tilde{M})$ be $\{\mathbf{e}_v\}_{v=1}^l$, and let $\left\{ \tilde{M}(\{\mathbf{e}_v\}) \right\}_{v=1}^{l-1}$ be linearly independent. Denoting the κ -th measurement result by $\omega_{1,\kappa}$, we can estimate $\text{tr} \rho_\theta \tilde{M}(\{\mathbf{e}_v\})$ by the relative frequency of observing \mathbf{e}_v , which is v -th component $\bar{\omega}_1^v$ of $\bar{\omega}_1 := \frac{1}{n} \sum_{\kappa=1}^n \omega_{1,\kappa}$. Let $\hat{\rho}$ be a solution to the system of linear equations $\text{tr} \hat{\rho} \tilde{M}(\{\mathbf{e}_v\}) = \bar{\omega}_1^v$ ($v = 1, \dots, l$), and \tilde{T}_n is defined by $\rho_{\tilde{T}_n} = \Pi(\hat{\rho})$, where Π is a properly defined projection. Also, if $\{\rho_\theta\}_{\theta \in \Theta}$ is a smooth submodel of quantum Gaussian model $\{\sigma_\eta\}$, we can compose $\tilde{\mathcal{E}}_n$ based on the estimator $\hat{\eta}_n$ of η by $\rho_{\tilde{T}_n} = \Pi(\sigma_{\hat{\eta}_n})$, with proerly defined projection Π .

Both of them has the following property. $\{\rho_\theta\}_{\theta \in \Theta}$ is a smooth submanifold of a larger quantum state model $\{\sigma_\eta\}$, where η has consistent estimator in the form of $\hat{\eta}_n = \frac{1}{n} \sum_{\kappa=1}^n \omega_{1,\kappa}$, where $\omega_{1,\kappa}$ is the data obtained by application of $\tilde{\mathbf{M}}$ on the κ -th sample. Suppose that $\eta = (\theta, \zeta)$, and $\rho_\theta = \sigma_{\theta, \zeta(\theta)}$. Moreover, we suppose that $\zeta(\theta)$ is uniformly continuous in θ . Then, $\tilde{T}_n := (\hat{\eta}_n^1, \dots, \hat{\eta}_n^m)$ satisfies the requirements.

As for the estimators, besides (1), we suppose $\mathcal{E}_n = \{\mathbf{M}^n, T_n\}$ satisfies (E) in Table 2 for all n . (E') is used to characterize lowerbound to the asymptotic cost. Observe that (E') \implies (E).

We define the *asymptotic quantum Cramer-Rao type bound* $C_\theta^Q(G_\theta, \mathcal{M})$ as $\lim_{n \rightarrow \infty} \inf \{n \text{Tr} G_\theta \text{MSE}_\theta[\mathcal{E}_n] ; \mathbf{M}^n \text{ in } \mathcal{H}^{\otimes n}, (1), (E)\}$. In the succeeding subsections, the following theorem will be proved. In the remaining of this subsection, some technical lemmas will be shown.

Theorem 1 *Suppose (M.1-3) hold. Then,*

$$C_\theta^Q(G_\theta, \mathcal{M}) = \lim_{n \rightarrow \infty} \inf \{n \text{Tr} G_\theta V_\theta[\mathcal{E}_{\theta,n}] ; \mathbf{M}^n \text{ in } \mathcal{H}^{\otimes n}, (2), (E')\}, \quad (3)$$

$$= \lim_{n \rightarrow \infty} \inf \{n \text{Tr} G_\theta V_\theta[\mathcal{E}_{\theta,n}] ; \mathbf{M}^n \text{ in } \mathcal{H}^{\otimes n}, (2), (E)\}. \quad (4)$$

Lemma 2 (E) and (M.1) imply the existence of $\partial_j E_\theta^{M^n} [T_n^i]$ and $\partial_j E_\theta^{M^n} [T_n^i] = \int T_n^i(\omega_n) \text{tr} \partial_j \rho_\theta M^n(d\omega_n)$.

Proof. Due to Lemma 3, this Lemma is equivalent to Proposition VI.2.2 of Holevo(1982). ■

Lemma 3 (M.1) implies that $\exists a_1 \exists a_2 \forall i, \forall \theta, \theta_0 \in \Theta$ and $|\theta^i - \theta_0^i| < a_2, \theta^j = \theta_0^j (j \neq i), \exists \Diamond_{i,\theta}$ such that $|\partial_i \rho_{\theta_0}| \leq \Diamond_{i,\theta}, \text{tr} \Diamond_{i,\theta} \leq a_1 < \infty$.

Proof. Since $\partial_i \rho_{\theta_0} = \partial_i \rho_{\theta - a_2 \mathbf{e}_i} + \int_{x=\theta - a_2 \mathbf{e}_i}^{\theta_0} \partial_i^2 \rho_x dx$, $\Diamond_{i,\theta} := |\partial_i \rho_{\theta - a_2 \mathbf{e}_i}| + \int_{x=\theta - a_2 \mathbf{e}_i}^{\theta + a_2 \mathbf{e}_i} |\partial_i^2 \rho_x| dx$, if exists in the sense of Bochner, satisfies requirement. This is true since $\|\partial_i^2 \rho_\theta\|_1$ is continuous in θ (hence, measurable and integrable over the finite interval). ■

Lemma 4 (E'), combined with (M.1), implies

$$\partial_j^{t_j} \partial_k^{t_k} E_\theta^{M^n} [T_n] = \int T_n(\omega_n) \text{tr} \partial_j^{t_j} \partial_k^{t_k} \rho_\theta^{\otimes n} M^n(d\omega_n) \quad (t_j, t_k \in \{0, 1\}), \quad (5)$$

$$|\text{Tr} G_\theta V_\theta [\mathcal{E}_n] - \text{Tr} G_{\theta'} V_{\theta'} [\mathcal{E}_n]| \leq (na_1 + 1) b_1 (a_{4,n})^2 \|\theta - \theta'\|, \quad (6)$$

$$\|E_\theta^{M^n} [T_n] - E_{\theta'}^{M^n} [T_n]\| \leq m^2 n a_{4,n} a_1 \|\theta - \theta'\|, \quad (7)$$

$$\|\partial_j E_\theta^{M^n} [T_n] - \partial_j E_{\theta'}^{M^n} [T_n]\| \leq m^2 n^2 a_{4,n} a_1^2 \|\theta - \theta'\|, \quad (8)$$

$$\lim_{\theta \rightarrow \theta_0} (B_{\theta_0} [\mathcal{E}_{\theta,n}])_j^i = \delta_j^i, \text{ where } \{\mathcal{E}_{\theta_0,n}\}_{\theta_0 \in \Theta} \text{ satisfies (2)}. \quad (9)$$

Proof. (E') implies $|\int T_n^i(\omega_n) \text{tr} \tau M^n(d\omega_n)| \leq |\int |T_n^i(\omega_n)| \text{tr} \tau M^n(d\omega_n)| \leq \|\tau\|_1 a_{4,n}$. Therefore, the map $\tau \rightarrow \int T_n^i(\omega_n) \text{tr} \tau M^n(d\omega_n)$ is a continuous linear functional, and is exchangeable with \lim . Therefore, the first two identities follow. To show (7), apply the mean value theorem to the function $\theta \rightarrow E_\theta^{M^n} [T_n]$. Due to (5), we obtain $|E_\theta^{M^n} [T_n^i] - E_{\theta'}^{M^n} [T_n^i]| \leq \sum_{j=1}^m |\int T_n^i(\omega_n) \text{tr} \partial_j \rho_{\theta_*}^{\otimes n} M^n(d\omega_n)| |\theta^j - \theta'^j|$. Therefore, due to (M.1) and Lemma 3, we have (7). (8) is shown similarly. To show (9), observe

$$\begin{aligned} |(B_{\theta_0} [\mathcal{E}_{\theta,n}])_j^i - \delta_j^i| &= |(B_{\theta_0} [\mathcal{E}_{\theta,n}])_j^i - (B_{\theta_0} [\mathcal{E}_{\theta_0,n}])_j^i| \\ &\leq |(B_{\theta_0} [\mathcal{E}_{\theta,n}])_j^i - (B_\theta [\mathcal{E}_{\theta,n}])_j^i| + |(B_\theta [\mathcal{E}_{\theta,n}])_j^i - (B_{\theta_0} [\mathcal{E}_{\theta_0,n}])_j^i| = |(B_{\theta_0} [\mathcal{E}_{\theta,n}])_j^i - (B_\theta [\mathcal{E}_{\theta,n}])_j^i|. \end{aligned}$$

Due to (8), we have (9). ■

3.2 LOWERBOUND AND (3)=(4)

First we prove that the RHS of (4) is a lowerbound to $C_\theta^Q(G_\theta, \mathcal{M})$. Define locally unbiased estimator $\mathcal{E}_{\theta,n} = \{M^n, T_{\theta,n}\}$ by $T_n = B_\theta [\mathcal{E}_n] (T_{\theta,n} - \theta) + E_\theta^{M^n} [T_n]$. Obviously,

$$n \text{Tr} G_\theta \text{MSE}_\theta [\mathcal{E}_n] \geq n \text{Tr} G_\theta V_\theta [\mathcal{E}_n] = n \text{Tr} G_\theta B_\theta [\mathcal{E}_n] V_\theta [\mathcal{E}_{\theta,n}] B_\theta [\mathcal{E}_n]^T,$$

and letting $n \rightarrow \infty$, we have our assertion due to (1).

Below, we prove (3)=(4). Since (E') implies (E), it suffices to show (3)≤(4). Suppose $\mathcal{E}_{\theta,n}$ satisfies (E) and (2). Let $S_{\theta,n}^L := T_{\theta,n}$ in $\|T_{\theta,n} - \theta\| \leq L$ -case and $S_{\theta,n}^L := \theta$ otherwise. Let $\mathcal{F}_{\theta,n}^L := \{M_\theta^n, T_{\theta,n}^L\}$, and

let $\mathcal{E}_{\theta,n}^L = \{M_{\theta,n}^n, T_{\theta,n}^L\}$ be a locally unbiased estimator with $T_{\theta,n}^L = B_{\theta} [\mathcal{F}_{\theta,n}^L]^{-1} (S_{\theta,n}^L - E_{\theta_0}^{M^n} [S_{\theta,n}^L]) + \theta$. Obviously, $\mathcal{E}_{\theta,n}^L$ satisfies (E'). Also, due to Lemma 2, Lemma 3, and (E), we have

$$\begin{aligned} \left| \left(B_{\theta} [\tilde{\mathcal{F}}_{\theta,n}^L] \right)_j^i - \delta_j^i \right| &= \left| \partial_i \int_{\|T_{\theta,n} - \theta\| > L} (T_{\theta,n}^j - \theta^j) P_{\theta}^{M_{\theta}^n} (d\omega_n) \right| = \left| \int_{\|T_{\theta,n} - \theta\| > L} (T_{\theta,n}^j - \theta^j) \text{tr} \partial_i \rho_{\theta}^{\otimes n} M_{\theta}^n (d\omega_n) \right| \\ &\leq \int_{\|T_{\theta,n} - \theta\| > L} \|T_{\theta,n} - \theta\| \text{tr} \diamond_{i,\theta,n}^{(1)} M_{\theta}^n (d\omega_n) \rightarrow 0 \quad (L \rightarrow \infty). \end{aligned}$$

Therefore, $\forall \varepsilon > 0 \exists L$,

$$\text{Tr} G_{\theta} V_{\theta} [\mathcal{E}_{\theta,n}] \geq \text{Tr} G_{\theta} V_{\theta} [\mathcal{F}_{\theta,n}^L] = \text{Tr} G_{\theta} B_{\theta} [\mathcal{F}_{\theta,n}^L] V_{\theta} [\mathcal{E}_{\theta,n}^L] B_{\theta} [\mathcal{F}_{\theta,n}^L]^T \geq \text{Tr} G_{\theta} V_{\theta} [\mathcal{E}_{\theta,n}^L] - \varepsilon.$$

Taking infimum of the both ends, we have (3) \leq (4).

3.3 ACHIEVABILITY

Based on $\{\mathcal{E}_{\theta,n_1}\}_{\theta \in \Theta} = \{M_{\theta,n_1}^{n_1}, T_{\theta,n_1}\}_{\theta \in \Theta}$ such that (2) and (E') with $n = n_1$ are satisfied, we construct a good estimator $\mathcal{E}_n^{n_1}$ with 2 steps in the following. Given $\rho_{\theta}^{\otimes n}$, invest $\rho_{\theta}^{\otimes n_0}$ to obtain the data $\vec{\omega}_1 := (\omega_{1,1}, \dots, \omega_{1,n_0})$, where $\omega_{1,i} \in \mathbb{R}^l$. Based on the data, we compute the estimator $\theta_0 = \tilde{T}_{n_0}(\vec{\omega}_1)$. Now, we divide $\rho_{\theta}^{\otimes n-n_0}$ into the ensembles each with n_1 copies. The number of ensemble, $\frac{n-n_0}{n_1}$, is denoted by n_2 . Here, n_0 and n_2 are chosen so that $n_0 = n_2^{3/4}$ is satisfied. We apply $M_{\theta_0}^{n_1}$ to each ensemble $\rho_{\theta}^{\otimes n_1}$, obtain the data $\omega_{2,1}, \dots, \omega_{2,n_2} (\in \mathbb{R}^{l_{n_1}})$ and compute

$$T_n^{n_1} := \frac{1}{n_2} \sum_{\kappa=1}^{n_2} T_{\theta_0,n_1}(\omega_{2,\kappa}). \quad (10)$$

The measurement defined above is denoted by $M^{n_1,n}$.

Lemma 5 Suppose that (M.1,3) hold. Suppose also that the family $\{\mathcal{E}_{\theta,n_1}\}_{\theta \in \Theta}$ satisfies (2) and (E') with $n = n_1$, $\forall \theta \in \Theta$. Then $\mathcal{E}_n^{n_1}$ constructed above satisfies $\lim_{n_2 \rightarrow \infty} n \text{Tr} G_{\theta} \text{MSE}_{\theta} [\mathcal{E}_n^{n_1}] \leq n_1 \overline{\lim}_{\theta_0 \rightarrow \theta} \text{Tr} G_{\theta} V_{\theta} [\mathcal{E}_{\theta_0,n_1}]$.

Proof. Applying mean value theorem to the function $\theta \rightarrow E_{\theta}^{M_{\theta_0}^{n_1}} [T_{\theta_0,n_1}^i]$, we have

$$E_{\theta}^{M_{\theta_0}^{n_1}} [T_{\theta_0,n_1}^i] = E_{\theta_0}^{M_{\theta_0}^{n_1}} [T_{\theta_0,n_1}^i] + \sum_{j=1}^m (\theta^j - \theta_0^j) \partial_j E_{\theta}^{M_{\theta_0}^{n_1}} [T_{\theta_0,n_1}^i] \Big|_{\theta=\theta_0} + \gamma_{\theta,\theta_0}^{n_1,i} = \theta_0^i + (\theta^i - \theta_0^i) + \gamma_{\theta,\theta_0}^{n_1,i} = \theta^i + \gamma_{\theta,\theta_0}^{n_1,i} \quad (11)$$

where $\gamma_{\theta,\theta_0}^{n_1,i}$ is the reminder term. With the help of (5) and (M.1),

$$|\gamma_{\theta,\theta_0}^{n_1,i}| = \frac{1}{2} \left| \sum_{j,k=1}^m (\theta^j - \theta_0^j) (\theta^k - \theta_0^k) \int T_{\theta_0,n_1}^i(\omega) \text{tr} \partial_j \partial_k \rho_{\theta'}^{\otimes n_1} M_{\theta_0}^{n_1} (d\omega) \right| \leq n_1^2 m^2 a_1^2 a_{4,n_1} \|\theta_0 - \theta\|^2, \quad (12)$$

where θ' lies between θ_0 and θ . Since MSE is the sum of the variance and square of the bias, we have

$$\begin{aligned} \text{Tr} G_{\theta} \text{MSE}_{\theta} [\mathcal{E}_n^{n_1} | \tilde{T}_{n_0} = \theta_0] &= \text{Tr} G_{\theta} V_{\theta} [\mathcal{E}_n^{n_1} | \tilde{T}_{n_0} = \theta_0] + \sum_{i,j=1}^m (G_{\theta})_{i,j} \gamma_{\theta,\theta_0}^{n_1,i} \gamma_{\theta,\theta_0}^{n_1,j} \\ &\leq \frac{1}{n_2} \text{Tr} G_{\theta} V_{\theta} [\mathcal{E}_{\theta_0,n_1}] + n_1^4 m^4 (a_1^2 a_{4,n_1})^2 \text{Tr} G_{\theta} \|\theta_0 - \theta\|^4. \end{aligned}$$

Taking average over \tilde{T}_{n_0} of the left most and the right most end,

$$\begin{aligned}
\lim_{n_2 \rightarrow \infty} n \text{Tr} G_\theta \text{MSE}_\theta [\mathcal{E}_n^{n_1}] &\leq \lim_{n_2 \rightarrow \infty} \left[\frac{n}{n_2} \text{E}_{\tilde{\theta}}^{\tilde{M}^{n_0}} \text{Tr} G_\theta \text{V}_\theta [\mathcal{E}_{\tilde{T}_{n_0}, n_1}] + n n_1^4 m^4 (a_1^2 a_{4, n_1})^2 \text{Tr} G_\theta \text{E}_{\tilde{\theta}}^{\tilde{M}^{n_0}} \|\tilde{T}_{n_0} - \theta\|^4 \right] \\
&\stackrel{(i)}{\leq} \lim_{n_2 \rightarrow \infty} \sup_{\theta_0: \|\theta_0 - \theta\| < \varepsilon} n_1 \text{Tr} G_\theta \text{V}_\theta [\mathcal{E}_{\theta_0, n_1}] + \frac{D_{\theta, 2}}{\varepsilon^4 n_0^2} \sup_{\theta_0 \in \mathbb{R}^m} \text{Tr} G_\theta \text{V}_\theta [\mathcal{E}_{\theta_0, n_1}] + n n_1^4 m^4 (a_1^2 a_{4, n_1})^2 \text{Tr} G_\theta \text{E}_{\tilde{\theta}}^{\tilde{M}^{n_0}} \|\tilde{T}_{n_0} - \theta\|^4 \\
&\stackrel{(ii)}{\leq} \lim_{n_2 \rightarrow \infty} \sup_{\theta_0: \|\theta_0 - \theta\| < \varepsilon} n_1 \text{Tr} G_\theta \text{V}_\theta [\mathcal{E}_{\theta_0, n_1}] + \frac{D_{\theta, 2}}{\varepsilon^4 n_2^{3/2}} (a_{4, n_1})^2 \text{Tr} G_\theta + \lim_{n_2 \rightarrow \infty} (n_2 n_1 + n_0) n_1^2 m^4 (a_1 a_{4, n_1})^2 \frac{D_{\theta, 2}}{n_2^{3/2}} \text{Tr} G_\theta \\
&= \sup_{\theta_0: \|\theta_0 - \theta\| < \varepsilon} n_1 \text{Tr} G_\theta \text{V}_\theta [\mathcal{E}_{\theta_0, n_1}].
\end{aligned}$$

Here (i) is due to $P_\theta^{\tilde{M}^{n_0}} \left\{ \|\tilde{T}_{n_0} - \theta\| \geq \varepsilon \right\} \leq \frac{D_{\theta, 2}}{\varepsilon^4 n_0^2}$ which follows from (M.3.2) and Chebyshev's inequality, and (ii) is due to (M.3.2). Since $\varepsilon > 0$ is arbitrary, the lemma holds. ■

Lemma 6 Suppose that (M.1-3) hold. Then $\{\mathcal{E}_n^{n_1}\}_{n=1}^\infty$ satisfies (E) and (1).

Proof. Observe $\|T_n^{n_1} - \theta\| \leq \|\tilde{T}_{n_0} - \theta\| + a_{4, n}$ holds. Since \tilde{T}_{n_0} satisfies (E) due to (M.3.1), $\{\mathcal{E}_n^{n_1}\}_{n=1}^\infty$ satisfies (E), also.

Observe

$$\begin{aligned}
\left| \text{E}_{\tilde{\theta}}^{\text{M}^{n_1, n}} [T_n^{n_1, j} - \theta_0^j] \right| &\leq \text{E}_{\tilde{\theta}}^{\tilde{M}^{n_0}} \left| \text{E}_{\tilde{\theta}}^{\text{M}_{\tilde{T}_{n_0}}^{n_1}} [T_{\tilde{T}_{n_0}, n_1}^j - \theta_0^j] \right| \stackrel{(i)}{\leq} \text{E}_{\tilde{\theta}}^{\tilde{M}^{n_0}} \left| \gamma_{\theta_0, \tilde{T}_{n_0}}^{n_1, j} \right| \\
&\stackrel{(ii)}{\leq} n_1^2 m^2 a_1 a_{4, n_1} \text{E}_{\tilde{\theta}}^{\tilde{M}^{n_0}} \|\tilde{T}_{n_0} - \theta_0\|^2 \stackrel{(iii)}{\leq} n_1^2 m^2 a_1 a_{4, n_1} \sqrt{\text{E}_{\tilde{\theta}}^{\tilde{M}^{n_0}} \|\tilde{T}_{n_0} - \theta_0\|^4} \stackrel{(iv)}{\rightarrow} 0.
\end{aligned}$$

Here, (i), (ii), (iii), and (iv) is due to (11), (12), concavity of \sqrt{x} , and (M.3.2), respectively. Therefore, $\text{E}_{\tilde{\theta}}^{\text{M}^{n_1, n}} [T_n^{n_1, j}] \rightarrow \theta_0^j$. $(B_{\theta_0}[\mathcal{E}_n])_j^i \rightarrow \delta_j^i$ is proved as follows. In Subsection 4.3 right after the statement of Lemma 11, we will prove

$$\partial_i \left(\text{E}_{\tilde{\theta}}^{\tilde{M}^{n_0}} \text{E}_{\tilde{\theta}}^{\text{M}_{\tilde{T}_{n_0}}^{n_1}} [T_{\tilde{T}_{n_0}, n_1}^j] \right)_{\theta=\theta_0} = \partial_i \left(\text{E}_{\tilde{\theta}}^{\tilde{M}^{n_0}} \text{E}_{\tilde{\theta}}^{\text{M}_{\tilde{T}_{n_0}}^{n_1}} [T_{\tilde{T}_{n_0}, n_1}^j] \right)_{\theta=\theta_0} + \text{E}_{\tilde{\theta}}^{\tilde{M}^{n_0}} \partial_i \text{E}_{\tilde{\theta}}^{\text{M}_{\tilde{T}_{n_0}}^{n_1}} [T_{\tilde{T}_{n_0}, n_1}^j]_{\theta=\theta_0}. \quad (13)$$

Defining $L_{\theta, i}^n := L_{\theta, i} \otimes \mathbf{1}^{\otimes n-1} + \mathbf{1} \otimes L_{\theta, i} \otimes \mathbf{1}^{\otimes n-2} + \dots + \mathbf{1}^{\otimes n-1} \otimes L_{\theta, i}$, we have $\partial_i \rho_\theta^{\otimes n} = \frac{1}{2} (L_{\theta, i}^n \rho_\theta^{\otimes n} + \rho_\theta^{\otimes n} L_{\theta, i}^n)$, $\text{tr} \rho_\theta^{\otimes n} (L_{\theta, i}^n)^2 = n \text{tr} \rho_\theta (L_{\theta, i})^2$, and

$$\partial_i \text{tr} \rho_\theta^{\otimes n} A = \text{tr} \partial_i \rho_\theta^{\otimes n} A = \Re \text{tr} \rho_\theta^{\otimes n} A L_{\theta, i}^n, \quad \forall A: \text{bounded Hermitian}, \quad (14)$$

where the first identity is due to the continuity of linear functional $X \rightarrow \text{tr} X A$ (e.g., Theorem II.7.2 of Holevo (1982)). (14), in combination with Schwartz's inequality, leads to $|\partial_i \text{tr} \rho_\theta^{\otimes n} X| \leq n \text{tr} \rho_\theta (L_{\theta, i})^2 \text{tr} \rho_\theta^{\otimes n} \text{tr} X^2$.

Observe $\left| \gamma_{\theta_0, \tilde{T}_{n_0}}^{n_1, j} \right| = \left| \text{E}_{\tilde{\theta}}^{\text{M}_{\tilde{T}_{n_0}}^{n_1}} [T_{\tilde{T}_{n_0}, n_1}^j] - \theta_0^j \right| \leq \|\tilde{T}_{n_0} - \theta_0^j\| + a_{4, n_1}$. Hence, due to (12), (M.3.1), and Lemma 2, we have $\partial_i \left[\text{E}_{\tilde{\theta}}^{\tilde{M}^{n_0}} \gamma_{\theta_0, \tilde{T}_{n_0}}^{n_1, j} \right]_{\theta=\theta_0} = \int \gamma_{\theta_0, \tilde{T}_{n_0}}^{n_1, j} \text{tr} \partial_i \rho_{\tilde{\theta}_0}^{\otimes n_0} \tilde{M}^{n_0} (d\vec{w}_1)$. Therefore, due to Theorem VI.2.1 of Holevo (1982)

and (14), the first term of (13) is evaluated as follows (they are used to show (i) below).

$$\begin{aligned} & \left| \partial_i \left(E_{\theta}^{\tilde{M}^{n_0}} E_{\theta_0}^{M_{\tilde{T}_{n_0}}^{n_1}} \left[T_{\tilde{T}_{n_0}, n_1}^j \right] \right) \right|_{\theta=\theta_0} = \left| \partial_i \left(E_{\theta}^{\tilde{M}^{n_0}} \left(\theta_0^j + \gamma_{\theta_0, \tilde{T}_{n_0}}^{n_1, j} \right) \right) \right|_{\theta=\theta_0} = \left| \partial_i \left[E_{\theta}^{\tilde{M}^{n_0}} \gamma_{\theta_0, \tilde{T}_{n_0}}^{n_1, j} \right] \right|_{\theta=\theta_0} \\ & \leq_{(i)} \sqrt{\text{tr} \rho_{\theta_0}^{\otimes n_0} \left(L_{\theta_0, i}^{n_0} \right)^2 E_{\theta_0}^{\tilde{M}^{n_0}} \left(\gamma_{\theta_0, \tilde{T}_{n_0}}^{n_1, j} \right)^2} \leq_{(ii)} \sqrt{n_0 \text{tr} \rho_{\theta_0} \left(L_{\theta_0, i} \right)^2 \cdot \frac{n_1^2 m^2 a_1 a_{4, n_1} D_{2, \theta_0}}{n_0^2}}, \end{aligned}$$

where (ii) is due to (12) and (M.3.2). Therefore, the first term vanishes as $n_0 \rightarrow \infty$. Due to (9) of Lemma 4, the second term converges to $\partial_i E_{\theta}^{M_{\theta_0}^{n_1}} \left[T_{\theta_0, n_1}^j \right]_{\theta=\theta_0} = \delta_j^i$, and (1) is proved. ■

Lemma 7 $\lim_{\theta_0 \rightarrow \theta} \inf \{ \text{Tr} G_{\theta} V_{\theta} [\mathcal{E}_{\theta_0, n}]; (2), (E') \} = \inf \{ \text{Tr} G_{\theta} V_{\theta} [\mathcal{E}_{\theta, n}]; (2), (E') \}$

Proof. Suppose the LHS is larger than the RHS (, denoted by A in the proof) by $2c > 0$. Then one can find a sequence $\{\theta_k\}$ such that $\lim_{k \rightarrow \infty} \inf \{ \text{Tr} G_{\theta} V_{\theta} [\mathcal{E}_{\theta_k, n}]; (2), (E') \} = A + 2c$. We prove this cannot occur.

Obviously, among those satisfying (2), (E'), one can find $\{\mathcal{E}_{\theta, n}\}_{\theta \in \Theta}$ such that $\text{Tr} G_{\theta} V_{\theta} [\mathcal{E}_{\theta, n}] \leq A + c$. Define $\mathcal{E}'_{\theta_k, n} := \left\{ M_{\theta}^n, T'_{\theta_k, n} \right\}$ by $T'_{\theta_k, n} := B_{\theta_k} [\mathcal{E}_{\theta, n}]^{-1} \left(T_{\theta, n} - E_{\theta_k}^{M_{\theta}^n} [T_{\theta, n}] \right) + \theta_k$. It is easy to verify $V_{\theta} [\mathcal{E}'_{\theta_k, n}] = B_{\theta_k} [\mathcal{E}_{\theta, n}]^{-1} \text{Tr} G_{\theta} V_{\theta} [\mathcal{E}_{\theta, n}] \left(B_{\theta_k} [\mathcal{E}_{\theta, n}]^{-1} \right)^T$ and that $\mathcal{E}'_{\theta_k, n}$ satisfies (2) and (E'). (Here note that $a_{4, n}$ has to be replaced by the other constant.) Therefore, due to (9) of Lemma 4, $\lim_{k \rightarrow \infty} \text{Tr} G_{\theta} V_{\theta} [\mathcal{E}'_{\theta_k, n}] = \text{Tr} G_{\theta} V_{\theta} [\mathcal{E}_{\theta, n}] \leq A + c < A + 2c = \lim_{k \rightarrow \infty} \inf \{ \text{Tr} G_{\theta} V_{\theta} [\mathcal{E}_{\theta_k, n}]; (2), (E') \}$. This is contradiction. ■

Due to Lemmas 5-7, we have ' \leq ' of (3) of Theorem 1.

3.4 On asymptotic normality of the estimator (10)

The estimator (10) is asymptotically normal. We prove the assertion in $m = 1$ -case, supposing that $\inf_{\theta_0 \in \mathbb{R}^m} V_{\theta} [\mathcal{E}_{\theta_0, n_1}]$ is not 0.

$$\begin{aligned} & \left| P_{\theta}^{M^{n_1, n}} \left\{ \sqrt{n} V_{\theta} [\mathcal{E}_{\theta, n_1}]^{-\frac{1}{2}} (T_n^{n_1} - \theta) \leq y \right\} - \Phi(y) \right| = \left| E_{\theta}^{\tilde{M}^{n_0}} P_{\theta}^{M_{\tilde{T}_{n_0}}^{n_1}} \left\{ \sqrt{n} V_{\theta} [\mathcal{E}_{\theta, n_1}]^{-\frac{1}{2}} (T_n^{n_1} - \theta) \leq y \right\} - \Phi(y) \right| \\ & \leq E_{\theta}^{\tilde{M}^{n_0}} \left| P_{\theta}^{M_{\tilde{T}_{n_0}}^{n_1}} \left\{ \sqrt{n} V_{\theta} [\mathcal{E}_{\tilde{T}_{n_0}, n_1}]^{-\frac{1}{2}} \left(T_n^{n_1} - \theta - \gamma_{\theta, \tilde{T}_{n_0}}^{n_1} \right) \leq y \right\} - \Phi(y) \right| \\ & + E_{\theta}^{\tilde{M}^{n_0}} \left| P_{\theta}^{M_{\tilde{T}_{n_0}}^{n_1}} \left\{ \sqrt{n} V_{\theta} [\mathcal{E}_{\theta, n_1}]^{-\frac{1}{2}} (T_n^{n_1} - \theta) \leq y \right\} - P_{\theta}^{M_{\tilde{T}_{n_0}}^{n_1}} \left\{ \sqrt{n} V_{\theta} [\mathcal{E}_{\tilde{T}_{n_0}, n_1}]^{-\frac{1}{2}} \left(T_n^{n_1} - \theta - \gamma_{\theta, \tilde{T}_{n_0}}^{n_1} \right) \leq y \right\} \right| \end{aligned}$$

Due to Berry-Esseen bound (Chapter 11 of DasGupta (2008)), the first term is upperbounded by $0.8 (a_{4, n_1})^3 n_2^{-\frac{1}{2}} E_{\theta}^{\tilde{M}^{n_0}} V_{\theta} [\mathcal{E}_{\theta, n_1}]$ and converges to 0 as $n_2 \rightarrow \infty$ since $\inf_{\theta_0 \in \mathbb{R}^m} V_{\theta} [\mathcal{E}_{\theta_0, n_1}] \neq 0$ by assumption. To evaluate the second term, we just have to consider the event such that $|\tilde{T}_{n_0} - \theta| < \varepsilon^{\frac{1}{2}} n^{-\frac{1}{4}}$, since the probability that this does not occur converges to 0 due to (M.3.2) and Chebyshev's inequality. Due to Lemma 7, we can suppose that $V_{\theta} [\mathcal{E}_{\theta_0, n_1}]$ is continuous in θ_0 at $\theta_0 = \theta$ without loss of generality. Therefore, $\left| V_{\theta} [\mathcal{E}_{\tilde{T}_{n_0}, n_1}]^{-1/2} - V_{\theta} [\mathcal{E}_{\theta, n_1}]^{-1/2} \right| < \varepsilon$ for large n . Let $c := y + n_1^2 m^2 a_1 a_{4, n_1}$. Since

$\left| \frac{1}{\sqrt{n}} \left(V_\theta \left[\mathcal{E}_{\tilde{T}_{n_0}, n_1} \right]^{\frac{1}{2}} - V_\theta \left[\mathcal{E}_{\theta, n_1} \right]^{\frac{1}{2}} \right) y + \gamma_{\theta, \tilde{T}_{n_0}}^{n_1} \right| \leq c\varepsilon n^{-\frac{1}{2}}$ due to (12), this is upperbounded by

$$\begin{aligned} & E_{\theta}^{\tilde{M}_{n_0}} P_{\theta}^{M_{\tilde{T}_{n_0}}^{n_1}} \left\{ y - c\varepsilon \leq \sqrt{n} V_\theta \left[\mathcal{E}_{\tilde{T}_{n_0}, n_1} \right]^{-\frac{1}{2}} \left(T_n^{n_1} - \theta - \gamma_{\theta, \tilde{T}_{n_0}}^{n_1} \right) \leq y + c\varepsilon \right\} \\ & \leq \Phi(y + c\varepsilon) - \Phi(y - c\varepsilon) + 0.8 (a_{4, n_1})^3 n_2^{-\frac{1}{2}} E_{\theta}^{\tilde{M}_{n_0}} V_\theta \left[\mathcal{E}_{\tilde{T}_{n_0}, n_1} \right]^{-\frac{3}{2}}. \end{aligned}$$

Here the inequality is due to Berry-Esseen bound. Letting $n \rightarrow \infty$ and $\varepsilon \rightarrow 0$, the last end converges to 0. After all, we have our assertion.

In $m \geq 2$ -case, the first term is evaluated using multi-dimensional version of Berry-Esseen bound (Chapter 11 of DasGupta (2008)). The second term is evaluated by analogous but more complicated analysis.

3.5 On logarithmic derivative and Fisher information

Hayashi and Matsumoto (1998) gives representation of $C^Q(G_\theta, \mathcal{M})$ using Fisher information $J_\theta^{M^n}$ of the classical statistical model $\{P_\theta^{M^n}\}_{\theta \in \Theta}$: $C^Q(G_\theta, \mathcal{M}) = \lim_{n \rightarrow \infty} \inf_{M^n} n \text{Tr } G_\theta J_\theta^{M^n}{}^{-1}$. They exploits the fact that the minimum variance of locally unbiased estimators equals $(J_\theta^{M^n})^{-1}$ and achieved by $T_{n, \theta}^j = \sum_{i=1}^m \left(J_\theta^{M^n}{}^{-1} \right)^{ij} l_{\theta, i}^{M^n} + \theta^j$, with $l_{\theta, i}^{M^n}$'s denoting the logarithmic derivative. Since their regularity conditions are different from ours, we examine here how far this statement holds in our setting. First, we define $l_{\theta, i}^{M^n}$ as the Radon-Nikodym derivative $d \text{tr } \partial_i \rho_\theta^{\otimes n} M^n / d \text{tr } \rho_\theta^{\otimes n} M^n$. Let $\mu^{M^n}(\Delta) := \text{tr } \sigma^{\otimes n} M^n(\Delta)$ ($\sigma > 0$) and $p_\theta^{M^n} := d(\text{tr } \rho_\theta^{\otimes n} M^n) / d\mu^{M^n}$, which exists since $\text{tr } \sigma^{\otimes n} M^n(\Delta) = 0$ implies $M^n(\Delta) = 0$. Since $\partial_i \int f p_\theta^{M^n} d\mu^{M^n} \leq (\sup |f|) \|\partial_i \rho_\theta^{\otimes n}\|_1$, there is L^1 function $\partial_i p_\theta^{M^n}$ such that $\partial_i \int f p_\theta^{M^n} d\mu^{M^n} = \int f \partial_i p_\theta^{M^n} d\mu^{M^n}$. Using this, $l_{\theta, i}^{M^n} = \partial_i p_\theta^{M^n} / p_\theta^{M^n}$, if the RHS is finite.

Lemma 8 *Suppose that (M.1) and (M.2) holds. Then $l_{\theta, i}^{M^n}$ exists and $\sup_{M^n} \mathbf{v}^T J_\theta^{M^n} \mathbf{v} J_\theta^{M^n}$ is finite. Also, if $\partial_i \rho_\theta \in \tau c(\mathcal{H})$ exists, and $\rho_\theta > 0$, $l_{\theta, i}^{M^n}$ exists.*

Proof. Suppose (M.2) holds. Since $\rho_\theta \geq 0$ and $M(\cdot) \geq 0$, $\text{tr } \rho_\theta^{\otimes n} M = 0$ means $\rho_\theta^{\otimes n} M = 0$. Therefore, due to (M.2), $\text{tr } \partial_i \rho_\theta^{\otimes n} M = \frac{1}{2} \left(\text{tr } L_{\theta, i}^n \rho_\theta^{\otimes n} M + \text{tr } M \rho_\theta^{\otimes n} L_{\theta, i}^n \right) = 0$. Therefore, $l_{\theta, i}^{M^n}$ exists. Define $l_{\theta, \mathbf{v}}^{M^n} := \sum_{i=1}^m v_i l_{\theta, i}^{M^n}$.

Let $\Delta_\iota := \left\{ \omega; \iota\varepsilon \leq \left(l_{\theta,\mathbf{v}}^{\mathbf{M}^n}(\omega) \right)^2 \leq (\iota+1)\varepsilon \right\}$, and denote by ω_ι the one satisfying $\left(l_{\theta,\mathbf{v}}^{\mathbf{M}^n}(\omega) \right)^2 = \iota\varepsilon$. Observe

$$\begin{aligned}
0 &\leq \sum_{i,j} v_i v_j \sum_{\iota} \text{tr} \rho_{\theta}^{\otimes n} \left\{ L_{\theta,i}^n - l_{\theta,i}^{\mathbf{M}^n}(\omega_\iota) \right\} M^n(\Delta_\iota) \left\{ L_{\theta,j}^n - l_{\theta,j}^{\mathbf{M}^n}(\omega_\iota) \right\} \\
&= \mathbf{v}^T J_{\theta}^{S,n} \mathbf{v} - 2 \sum_j \sum_{\iota} v_j l_{\theta,\mathbf{v}}^{\mathbf{M}^n}(\omega_\iota) \Re \text{tr} \rho_{\theta}^{\otimes n} L_{\theta,j}^n M^n(\Delta_\iota) + \sum_{\iota} \left(l_{\theta,\mathbf{v}}^{\mathbf{M}^n}(\omega_\iota) \right)^2 \text{tr} \rho_{\theta}^{\otimes n} M^n(\Delta_\iota) \\
&= \mathbf{v}^T J_{\theta}^{S,n} \mathbf{v} - 2 \sum_{\iota} l_{\theta,\mathbf{v}}^{\mathbf{M}^n}(\omega_\iota) \int_{\Delta_\iota} l_{\theta,\mathbf{v}}^{\mathbf{M}^n}(\omega) \text{tr} \rho_{\theta}^{\otimes n} M^n(d\omega) + \sum_{\iota} \left(l_{\theta,\mathbf{v}}^{\mathbf{M}^n}(\omega_\iota) \right)^2 \text{tr} \rho_{\theta}^{\otimes n} M^n(\Delta_\iota) \\
&= \mathbf{v}^T J_{\theta}^{S,n} \mathbf{v} - \sum_{\iota} \left(l_{\theta,\mathbf{v}}^{\mathbf{M}^n}(\omega_\iota) \right)^2 \text{tr} \rho_{\theta}^{\otimes n} M^n(\Delta_\iota) - 2 \sum_{\iota} l_{\theta,\mathbf{v}}^{\mathbf{M}^n}(\omega_\iota) \left(\int_{\Delta_\iota} l_{\theta,\mathbf{v}}^{\mathbf{M}^n}(\omega) \text{tr} \rho_{\theta}^{\otimes n} M^n(d\omega) - l_{\theta,\mathbf{v}}^{\mathbf{M}^n}(\omega_\iota) \text{tr} \rho_{\theta}^{\otimes n} M^n(\Delta_\iota) \right) \\
&\leq \mathbf{v}^T J_{\theta}^{S,n} \mathbf{v} - \int \left(l_{\theta,\mathbf{v}}^{\mathbf{M}^n}(\omega) \right)^2 \rho_{\theta}^{\otimes n} M^n(d\omega) + \varepsilon + 2 \sum_{\iota} \int_{\Delta_\iota} \left| l_{\theta,\mathbf{v}}^{\mathbf{M}^n}(\omega_\iota) \right| \frac{\varepsilon}{\left| l_{\theta,\mathbf{v}}^{\mathbf{M}^n}(\omega) + l_{\theta,\mathbf{v}}^{\mathbf{M}^n}(\omega_\iota) \right|} \text{tr} \rho_{\theta}^{\otimes n} M^n(d\omega) \\
&\leq \mathbf{v}^T J_{\theta}^{S,n} \mathbf{v} - \int \left(l_{\theta,\mathbf{v}}^{\mathbf{M}^n}(\omega) \right)^2 \rho_{\theta}^{\otimes n} M^n(d\omega) + \varepsilon + 2\varepsilon
\end{aligned}$$

which, with $\varepsilon \rightarrow 0$, implies $\mathbf{v}^T J_{\theta}^{\mathbf{M}^n} \mathbf{v} \leq \mathbf{v}^T J_{\theta}^{S,n} \mathbf{v} < \infty$. Also, suppose $\rho_{\theta} > 0$. Then, $\text{tr} \rho_{\theta}^{\otimes n} M = 0$ means $M = 0$ and $\text{tr} \partial_i \rho_{\theta}^{\otimes n} M = 0$. Therefore, the second assertion is proved. ■

The RHS of (3) and (4) is lowerbounded by $\inf_{\mathbf{M}^n} \text{Tr} G_{\theta} \left(J_{\theta}^{\mathbf{M}^n} \right)^{-1}$, due to Schwartz's inequality. Achievability, in fact, also holds. Define $\mathcal{E}_{\theta,n}^L := \left\{ \mathbf{M}_{\theta}^n, T_{n,\theta}^L \right\}$ by $l_{\theta,i}^{\mathbf{M}^n,L} := \chi_{\{\omega_n; \|l_{\theta,i}^{\mathbf{M}^n}\| \leq L\}} l_{\theta,i}^{\mathbf{M}^n}$, $\left(J_{\theta}^{\mathbf{M}^n,L} \right)_{i,j} := \mathbb{E}_{\theta}^{\mathbf{M}^n} l_{\theta,i}^{\mathbf{M}^n,L} l_{\theta,j}^{\mathbf{M}^n,L}$, and $T_{n,\theta}^{L,j} := \sum_{i=1}^m \left[\left(J_{\theta}^{\mathbf{M}^n,L} \right)^{-1} \right]^{ij} l_{\theta,i}^{\mathbf{M}^n,L} + \theta^j$. Obviously, $\mathcal{E}_{\theta,n}^L$ satisfies (E'). Therefore, due to Lemma 4,

$$\partial_j \mathbb{E}_{\theta}^{\mathbf{M}^n} l_{\theta,i}^{\mathbf{M}^n,L} = \int l_{\theta,i}^{\mathbf{M}^n,L} \text{tr} \partial_j \rho_{\theta}^{\otimes n} M^n(d\omega_n) = \int l_{\theta,i}^{\mathbf{M}^n,L} l_{\theta,j}^{\mathbf{M}^n} \text{tr} \rho_{\theta}^{\otimes n} M^n(d\omega_n) = \int l_{\theta,i}^{\mathbf{M}^n,L} l_{\theta,j}^{\mathbf{M}^n,L} \text{tr} \rho_{\theta} M^n(d\omega_n) = J_{\theta}^{\mathbf{M}^n,L}.$$

Therefore, $\mathcal{E}_{\theta,n}^L$ also satisfies (1). Also, $\text{Tr} G_{\theta} \mathbf{V}_{\theta} \left[\mathcal{E}_{\theta,n}^L \right] = \text{Tr} G_{\theta} \left(J_{\theta}^{\mathbf{M}^n,L} \right)^{-1}$. Hence, it remains to show $\lim_{L \rightarrow \infty} \mathbf{v}^T J_{\theta}^{\mathbf{M}^n,L} \mathbf{v} = \mathbf{v}^T J_{\theta}^{\mathbf{M}^n} \mathbf{v}$, $\forall \mathbf{v}$. This is true since $\mathbf{v}^T J_{\theta}^{\mathbf{M}^n} \mathbf{v} = \int \left(\sum_{i=1}^m v^i l_{\theta,i}^{\mathbf{M}^n} \right)^2 P_{\theta}^{\mathbf{M}^n}(d\omega_n) < \infty$.

Therefore, logarithmic derivative and the Fisher information can be used to represent $C^Q(G_{\theta}, \mathcal{M})$. However, it is not possible to show their chain rule, which is at the heart of the argument for one-way semi-classical setting in Hayashi and Matsumoto(1998). Therefore, in the next subsection, we use somewhat different method to prove the asymptotic Cramer-Rao type bound in the semi-classical setting.

1 4. SEMI-CLASSICAL MEASUREMENT

4.1 DEFINITIONS, REGULARITY CONDITIONS, AND MAIN THEOREM

An important subclass of measurements is *semi-classical* measurements, which are composed adoptively in R_n ($< \infty$) rounds. At each round, we measure each sample separately, and the measurements of the r -th round depend on the previously obtained data. We denote by $z_{r,\kappa} (\in \mathbb{R}^l)$ the data obtained at the r -th round from κ -th sample, and \mathbf{z}_r is the data $(z_{1,1}, z_{1,2}, \dots, z_{r,n})$ obtained up to the r -th round. The measurement

acting in the r -th round on κ -th sample is denoted by $M_{r,\kappa}^{\mathbf{z}_{r-1}}$. Without loss of generality, we suppose that in the first round the measurement is chosen deterministically. Rigorous mathematical description of such a process is given in the following subsection.

Define $\mathbf{z}_{r,\kappa}^\downarrow := (\mathbf{z}_{r-1}, z_{r,1}, \dots, z_{r,\kappa})$, and $\mathbf{z}_{r,\kappa}^\uparrow := (z_{r,\kappa}, z_{r,\kappa+1}, \dots, z_{R_n,n})$. \mathfrak{B}_r , $\mathfrak{B}_{r,\kappa}$, $\mathfrak{B}_{r,\kappa}^\downarrow$ and $\mathfrak{B}_{r,\kappa}^\uparrow$ is the totality of Borel sets over the space where \mathbf{z}_r , $z_{r,\kappa}$, $\mathbf{z}_{r,\kappa}^\downarrow$, and $\mathbf{z}_{r,\kappa}^\uparrow$ is living in, respectively. The instrument corresponding to successive application of $M_{1,1}$, $M_{1,2}$, \dots , $M_{r,\kappa}^{\mathbf{z}_{r-1}}$ and $M_{r,\kappa}^{\mathbf{z}_{r-1}}$, $M_{r,\kappa+1}^{\mathbf{z}_{r-1}}$, \dots , $M_{R_n,n}^{\mathbf{z}_{R_n-1}}$ is denoted by $M_{r,\kappa}^\downarrow$ and $M_{r,\kappa}^{\uparrow \mathbf{z}_{r-1}}$, respectively. Note that they depend on n , although we do not denote the fact explicitly for the sake of simplicity. Note also that R_n is arbitrary but finite.

Note that in other literatures such as Hayashi and Matsumoto (1999), the term ‘semi-classical measurement’ refers to more restricted class of measurement, which is called *one-way semi-classical* measurement in this paper. The restriction is that in r -th round, we measure r -th sample only (Hence, $R_n = n$).

The *asymptotic semi-classical Cramer-Rao* type bound $C_\theta(G_\theta, \mathcal{M})$ is defined by

$$C_\theta(G_\theta, \mathcal{M}) := \overline{\lim}_{n \rightarrow \infty} \inf \{n \text{Tr } G_\theta \text{MSE}_\theta[\mathcal{E}_n] ; \mathbf{M}^n \text{ in } \mathcal{H}^{\otimes n}, \text{ semi-classical, (1), (E)}\}.$$

Theorem 9 Suppose (M.1-3) hold. Then,

$$\begin{aligned} C_\theta(G_\theta, \mathcal{M}) &= \inf \{ \text{Tr } G_\theta V_\theta[\mathcal{E}_{\theta_0,1}] ; \mathbf{M}^1 \text{ in } \mathcal{H}, (2), (E') \text{ with } n = 1 \}, \\ &= \inf \{ \text{Tr } G_\theta V_\theta[\mathcal{E}_{\theta_0,1}] ; \mathbf{M}^1 \text{ in } \mathcal{H}, (2), (E) \text{ with } n = 1 \}. \end{aligned}$$

4.2 ADAPTIVE MEASUREMENT

In this subsection, we give mathematically rigorous account on a composite measurement NM of measurement M followed by N^ω , where N^ω is composed depending on the data $\omega \in \mathbb{R}^l$ from M. More specifically, $\omega \rightarrow N^\omega[\Delta']$ can be approximated by a sequence of simple functions except for $\omega \in \Delta$ where $M(\Delta) = 0$, so that the function is strongly measurable with respect to P_ρ^M for any ρ . We show this composite NM can be described using an instrument. (The contents of this subsection should be well-known to specialists of the field of measurement theory. The author, however, could not find a proper reference.)

The key fact is Theorem 4.5 of Ozawa (1985), or that there is a family of density operators $\{\rho_\omega^M\}_{\omega \in \mathbb{R}^l}$ (*a posteriori states*) with $\int_{\omega \in \Delta} \text{tr } A \rho_\omega^M P_\rho^M(d\omega) = \text{tr } A M[\Delta](\rho)$ ($\forall A \in \mathcal{B}(\mathcal{H})$). Since $\mathcal{B}(\mathcal{H})$ is the dual of $\tau\mathcal{C}(\mathcal{H})$ with the pairing $\langle \rho, A \rangle := \text{tr } \rho A$ (Theorem II.7.2 of Holevo (1982)), Ozawa’s statement is equivalent to the weak measurability and the existence of Pettis integral of the function $\omega \rightarrow \rho_\omega^M$. As summarized in the end of Subsection 2.1, $\omega \rightarrow \rho_\omega^M$ in fact is strongly measurable. Also, since $\int_{\omega \in \Delta} \|\rho_\omega^M\|_1 P_\rho^M(d\omega) = 1 < \infty$, the Bochner integral $\int_{\omega \in \Delta} \rho_\omega^M P_\rho^M(d\omega) = M[\Delta](\rho)$ is convergent.

First, we show P_ρ^{NM} is well-defined. Since $\omega \rightarrow N^\omega[\Delta']$ and $\omega \rightarrow \rho_\omega^M$ are strongly measurable, they can be approximated by simple functions. Therefore, $\omega \rightarrow \text{tr } N^\omega[\Delta'] \rho_\omega^M$ is a measurable function for any $\Delta' \in$

$\mathfrak{B}(\mathbb{R}^l)$, and $P_\rho^{\text{NM}}(\Delta \times \Delta') := \int_{\omega \in \Delta} \text{tr } N^\omega[\Delta'] \rho_\omega^{\text{M}} P_\rho^{\text{M}}(d\omega)$ is well-defined and σ -additive. Therefore, P_ρ^{NM} can be extended to $\mathfrak{B}(\mathbb{R}^l \times \mathbb{R}^l)$ due to Hopf's extension theorem. Moreover, with $\tilde{\Delta}_\omega := \{\omega'; (\omega, \omega') \in \tilde{\Delta}\}$, $\int_\omega \text{tr } N^\omega[\tilde{\Delta}_\omega] \rho_\omega^{\text{M}} P_\rho^{\text{M}}(d\omega)$ exists and equals $P_\rho^{\text{NM}}\{\tilde{\Delta}\}$ for any Borel set $\tilde{\Delta}$; Let \mathfrak{D} be the totality of $\tilde{\Delta}$ such that the assertion is true. Obviously, \mathfrak{D} is a Dynkin system, and contains cylinder sets. Therefore, due to Dynkin's lemma, $\mathfrak{D} = \mathfrak{B}(\mathbb{R}^l \times \mathbb{R}^l)$.

Next, we show that $\rho_{\tilde{\Delta}}^{\text{NM}}$ is well-defined. Since $\int_\omega \|N^\omega[\tilde{\Delta}_\omega] \rho_\omega^{\text{M}}\|_1 P_\rho^{\text{M}}(d\omega) \leq 1$, the Bochner integral $\text{NM}[\tilde{\Delta}](\rho) := \int_\omega N^\omega[\tilde{\Delta}_\omega] \rho_\omega^{\text{M}} P_\rho^{\text{M}}(d\omega)$ is convergent. Also, its trace equals $P_\rho^{\text{NM}}\{\tilde{\Delta}\}$, since tr and \int can be exchanged due to Fubini's theorem.

In addition, $\rho \rightarrow \text{NM}[\tilde{\Delta}](\rho)$ is affine and completely positive, as is proved in the following. Observe Bochner integral $\int_\omega N^\omega[\tilde{\Delta}_\omega] \rho_\omega^{\text{M}}(d\omega)$ in $\mathcal{B}(\tau c(\mathcal{H}))$ is well-defined in terms of $\|\cdot\|_{cb}$, due to $\int_\omega \|N^\omega[\tilde{\Delta}_\omega]\|_{cb} P_\rho^{\text{M}}(d\omega) \leq 1$. Therefore, there exist sequences of families $\{N_j^{(k)}\}_j$ and $\{\Delta_j^{(k)}\}_j$ ($k = 1, \dots, \infty$) of completely positive maps and Borel sets, such that for any ρ

$$\begin{aligned} \left\| \text{NM}[\tilde{\Delta}](\rho) - \sum_j N_j^{(k)} \text{M}[\Delta_j^{(k)}](\rho) \right\|_1 &= \left\| \int_\omega \left\{ N^\omega[\tilde{\Delta}_\omega] - \sum_j N_j^{(k)} \chi_{\Delta_j^{(k)}} \right\} \rho_\omega^{\text{M}} P_\rho^{\text{M}}(d\omega) \right\|_1 \\ &\leq \int_\omega \left\| \left\{ N^\omega[\tilde{\Delta}_\omega] - \sum_j N_j^{(k)} \chi_{\Delta_j^{(k)}} \right\} \rho_\omega^{\text{M}} \right\|_1 P_\rho^{\text{M}}(d\omega) \leq \int_\omega \left\| N^\omega[\tilde{\Delta}_\omega] - \sum_j N_j^{(k)} \chi_{\Delta_j^{(k)}} \right\|_{cb} P_\rho^{\text{M}}(d\omega) \rightarrow 0, \end{aligned}$$

as $k \rightarrow \infty$. Since $\sum_j N_j^{(k)} \text{M}[\Delta_j^{(k)}]$ is affine and completely positive, we have our assertion.

Finally, $\tilde{\Delta} \rightarrow \text{NM}[\tilde{\Delta}]$ is an instrument. Obviously, $\text{tr } \text{NM}[\mathbb{R}^l \times \mathbb{R}^l](\rho) = 1$. Also,

$$\begin{aligned} \text{NM}\left[\bigcup_{j=1}^{\infty} \tilde{\Delta}_j\right](\rho) &= \int_\omega N^\omega\left[\bigcup_{j=1}^{\infty} \tilde{\Delta}_{j,\omega}\right] \rho_\omega^{\text{M}} P_\rho^{\text{M}}(d\omega) = \int_\omega \sum_{j=1}^{\infty} N^\omega[\tilde{\Delta}_{j,\omega}] \rho_\omega^{\text{M}} P_\rho^{\text{M}}(d\omega) \\ &= \sum_{j=1}^{\infty} \int_\omega N^\omega[\tilde{\Delta}_{j,\omega}] \rho_\omega^{\text{M}} P_\rho^{\text{M}}(d\omega) = \sum_{j=1}^{\infty} \text{NM}[\tilde{\Delta}_j](\rho), \end{aligned}$$

where the third identity is due to Fubini's theorem of Bochner integral. Therefore, $\tilde{\Delta} \rightarrow \text{NM}[\tilde{\Delta}]$ is σ -additive in terms of strong operator topology in $\mathcal{B}(\tau c(\mathcal{H}))$.

4.3 LEIBNIZ RULE

In this subsection and the next, so far as no confusion is likely to arise, we denote $P_\theta^{\text{M}_\theta^n}$ and $E_\theta^{\text{M}_\theta^n}$ by P_θ and E_θ , respectively, where M_θ^n is a semi-classical measurement.

Lemma 10 *Suppose T_n satisfies (E'). Suppose also (a) $\rho_\theta > 0$, $\forall \theta \in \Theta$, or (b) the estimator is one-way semi-classical. Then, $\exists \Delta \in \mathfrak{B}_n$ s.t. $M^n(\Delta) = 0$ and $E_\theta[T_n | \mathfrak{B}_{r,\kappa}^\downarrow](\mathbf{z}_{r,\kappa}^\downarrow)$ is continuous in θ for $\forall r \forall \kappa$, if $\mathbf{z}_n \notin \Delta$.*

Proof. The case (b) is due to the fact that $M_{r,r}^{\uparrow \mathbf{z}_{r-1}}$ acts on $\rho_\theta^{\otimes(n-r+1)}$:

$$E_\theta[T_n | \mathfrak{B}_{r,\kappa}^\downarrow](\mathbf{z}_{r,\kappa}^\downarrow) = \int T_n(\mathbf{z}_{r,r}^\uparrow, \mathbf{z}_{r-1}) \text{tr } \rho_\theta^{\otimes(n-r+1)} M_{r,r}^{\uparrow \mathbf{z}_{r-1}}(d\mathbf{z}_{r,r}^\uparrow). \text{ Here, by abuse of notation, } M_{r,r}^{\uparrow \mathbf{z}_{r-1}}(\Delta) \in$$

$\mathcal{B}(\mathcal{H}^{\otimes n-r+1})$ in case of one-way semi-classical measurements.

The case (a) is proved as follows. Let $g_{\theta, \theta'}(\mathbf{z}_{r, \kappa}^\downarrow) := \mathbb{E}_{\theta'}[T_n | \mathfrak{B}_{r, \kappa}^\downarrow](\mathbf{z}_{r, \kappa}^\downarrow) - \mathbb{E}_\theta[T_n | \mathfrak{B}_{r, \kappa}^\downarrow](\mathbf{z}_{r, \kappa}^\downarrow)$. Observe $\int_{\mathbf{z}_{r, \kappa}^\downarrow \in \Delta} \mathbb{E}_\theta[T_n | \mathfrak{B}_{r, \kappa}^\downarrow](\mathbf{z}_{r, \kappa}^\downarrow) P_\theta^{\mathbf{M}_{r, \kappa}^\downarrow}(d\mathbf{z}_{r, \kappa}^\downarrow)$, which equals $\int_{\mathbf{z}_{r, \kappa}^\downarrow \in \Delta} T_n(\mathbf{z}_{r, \kappa}^\downarrow, \mathbf{z}_{r, \kappa+1}^\uparrow) \text{tr} \rho_\theta^{\otimes n} M^n(d\mathbf{z}_{r, \kappa}^\downarrow d\mathbf{z}_{r, \kappa+1}^\uparrow)$, is continuous in θ for any Borel set Δ , due to (7) of Lemma 4. Also, observe

$$\begin{aligned} & \int_{\mathbf{z}_{r, \kappa}^\downarrow \in \Delta} g_{\theta, \theta'}(\mathbf{z}_{r, \kappa}^\downarrow) P_\theta^{\mathbf{M}_{r, \kappa}^\downarrow}(d\mathbf{z}_{r, \kappa}^\downarrow) \\ &= \int_{\mathbf{z}_{r, \kappa}^\downarrow \in \Delta} \left\{ \mathbb{E}_{\theta'}[T_n | \mathfrak{B}_{r, \kappa}^\downarrow](\mathbf{z}_{r, \kappa}^\downarrow) P_{\theta'}^{\mathbf{M}_{r, \kappa}^\downarrow}(d\mathbf{z}_{r, \kappa}^\downarrow) - \mathbb{E}_\theta[T_n | \mathfrak{B}_{r, \kappa}^\downarrow](\mathbf{z}_{r, \kappa}^\downarrow) P_\theta^{\mathbf{M}_{r, \kappa}^\downarrow}(d\mathbf{z}_{r, \kappa}^\downarrow) \right\} \\ &+ \int_{\mathbf{z}_{r, \kappa}^\downarrow \in \Delta} \mathbb{E}_{\theta'}[T_n | \mathfrak{B}_{r, \kappa}^\downarrow](\mathbf{z}_{r, \kappa}^\downarrow) \left\{ P_\theta^{\mathbf{M}_{r, \kappa}^\downarrow}(d\mathbf{z}_{r, \kappa}^\downarrow) - P_{\theta'}^{\mathbf{M}_{r, \kappa}^\downarrow}(d\mathbf{z}_{r, \kappa}^\downarrow) \right\}. \\ &= \int_{\mathbf{z}_{r, \kappa}^\downarrow \in \Delta} T_n(\mathbf{z}_{r, \kappa}^\downarrow, \mathbf{z}_{r, \kappa+1}^\uparrow) \text{tr}(\rho_{\theta'}^{\otimes n} - \rho_\theta^{\otimes n}) M^n(d\mathbf{z}_{r, \kappa}^\downarrow d\mathbf{z}_{r, \kappa+1}^\uparrow) \\ &+ \int_{\mathbf{z}_{r, \kappa}^\downarrow \in \Delta} \mathbb{E}_{\theta'}[T_n | \mathfrak{B}_{r, \kappa}^\downarrow](\mathbf{z}_{r, \kappa}^\downarrow) \text{tr}(\rho_{\theta'}^{\otimes n} - \rho_\theta^{\otimes n}) M_{r, \kappa}^\downarrow(d\mathbf{z}_{r, \kappa}^\downarrow d\mathbf{z}_{r, \kappa+1}^\uparrow) \end{aligned}$$

Tending $\theta' \rightarrow \theta$, the last end converges to 0, and so does the left most side. Hence, due to bounded convergence theorem, we have $\int_{\mathbf{z}_{r, \kappa}^\downarrow \in \Delta} [\lim_{\theta' \rightarrow \theta} g_{\theta, \theta'}(\mathbf{z}_{r, \kappa}^\downarrow)] P_\theta^{\mathbf{M}_{r, \kappa}^\downarrow}(d\mathbf{z}_{r, \kappa}^\downarrow) = 0$ for any Borel set Δ . Therefore, $g_{\theta, \theta'}(\mathbf{z}_{r, \kappa}^\downarrow) = 0$ for P_θ -a.e. Since $P_\theta(\Delta) = \text{tr} \rho_\theta M(\Delta) = 0 \Rightarrow \rho_\theta M(\Delta) = 0$ due to $\rho_\theta > 0$, the proof is complete. ■

Lemma 11 Suppose (E') is satisfied. Suppose also (a) $\rho_\theta > 0, \forall \theta \in \Theta$, or (b) the estimator is one-way semi-classical. Then, we have Leibniz rule:

$$\partial_i \mathbb{E}_\theta[T_n] |_{\theta=\theta_0} = [\partial_i \mathbb{E}_\theta \mathbb{E}_{\theta_0}[T_n | \mathfrak{B}_{r, \kappa}^\downarrow] + \partial_i \mathbb{E}_{\theta_0} \mathbb{E}_\theta[T_n | \mathfrak{B}_{r, \kappa}^\downarrow]]_{\theta=\theta_0}. \quad (15)$$

(13) is a special case of (15). To see this, observe that the estimator (10) is viewed as semi-classical considering the quantum statistical model $\{\rho_\theta^{\otimes n_1}\}_{\theta \in \Theta}$.

Proof.

$$\begin{aligned} \partial_i \mathbb{E}_\theta[T_n] |_{\theta=\theta_0} &= \partial_i \mathbb{E}_\theta \mathbb{E}_{\theta_0}[T_n | \mathfrak{B}_{r, \kappa}^\downarrow] |_{\theta=\theta_0} \\ &= \lim_{\theta' \rightarrow \theta_0} \left[\frac{\mathbb{E}_{\theta'} \mathbb{E}_{\theta_0}[T_n | \mathfrak{B}_{r, \kappa}^\downarrow] - \mathbb{E}_{\theta_0} \mathbb{E}_{\theta_0}[T_n | \mathfrak{B}_{r, \kappa}^\downarrow]}{\|\theta' - \theta_0\|} + \frac{\mathbb{E}_{\theta'}[\mathbb{E}_{\theta'}[T_n | \mathfrak{B}_{r, \kappa}^\downarrow] - \mathbb{E}_{\theta_0}[T_n | \mathfrak{B}_{r, \kappa}^\downarrow]]}{\|\theta' - \theta_0\|} \right], \quad (16) \end{aligned}$$

where the convention is that $\theta'^j = \theta_0^j$ ($j \neq i$). The first term converges to $\partial_i \mathbb{E}_\theta \mathbb{E}_{\theta_0}[T_n | \mathfrak{B}_{r, \kappa}^\downarrow]_{\theta=\theta_0}$, due to $|\mathbb{E}_{\theta_0}[T_n | \mathfrak{B}_{r, \kappa}^\downarrow]| < a_{4, n}$ and Lemma 4. Observe that the second term should converge due to the convergence

of (16) and the first term. Moreover,

$$\begin{aligned}
& \left| \frac{\mathbb{E}_{\theta'} [\mathbb{E}_{\theta'} [T_n^j | \mathfrak{B}_{r,\kappa}^\downarrow] - \mathbb{E}_{\theta_0} [T_n^j | \mathfrak{B}_{r,\kappa}^\downarrow]]}{\|\theta' - \theta_0\|} - \frac{\mathbb{E}_{\theta_0} [\mathbb{E}_{\theta'} [T_n^j | \mathfrak{B}_{r,\kappa}^\downarrow] - \mathbb{E}_{\theta_0} [T_n^j | \mathfrak{B}_{r,\kappa}^\downarrow]]}{\|\theta' - \theta_0\|} \right| \\
&= \left| \int (\mathbb{E}_{\theta'} [T_n^j | \mathfrak{B}_{r,\kappa}^\downarrow] - \mathbb{E}_{\theta_0} [T_n^j | \mathfrak{B}_{r,\kappa}^\downarrow]) \operatorname{tr} \left(\frac{\rho_{\theta'}^{\otimes n} - \rho_{\theta_0}^{\otimes n}}{\|\theta' - \theta_0\|} \right) M_{\theta_0, r, \kappa}^\downarrow (d\mathbf{z}_{r,\kappa}^\downarrow) \right| \\
&\stackrel{(i)}{=} \left| \frac{\partial}{\partial \theta^i} \left[\int (\mathbb{E}_{\theta'} [T_n^j | \mathfrak{B}_{r,\kappa}^\downarrow] - \mathbb{E}_{\theta_0} [T_n^j | \mathfrak{B}_{r,\kappa}^\downarrow]) \operatorname{tr} \rho_{\theta}^{\otimes n} M_{\theta_0, r, \kappa}^\downarrow (d\mathbf{z}_{r,\kappa}^\downarrow) \right]_{\theta=\bar{\theta}} \right| \\
&\stackrel{(ii)}{=} \left| \int (\mathbb{E}_{\theta'} [T_n^j | \mathfrak{B}_{r,\kappa}^\downarrow] - \mathbb{E}_{\theta_0} [T_n^j | \mathfrak{B}_{r,\kappa}^\downarrow]) \operatorname{tr} \partial_i \rho_{\bar{\theta}}^{\otimes n} M_{\theta_0, r, \kappa}^\downarrow (d\mathbf{z}_{r,\kappa}^\downarrow) \right| \\
&\leq \int |\mathbb{E}_{\theta'} [T_n^j | \mathfrak{B}_{r,\kappa}^\downarrow] - \mathbb{E}_{\theta_0} [T_n^j | \mathfrak{B}_{r,\kappa}^\downarrow]| \operatorname{tr} \diamond_{i, \theta_0, n}^{(1)} M_{\theta_0, r, \kappa}^\downarrow (d\mathbf{z}_{r,\kappa}^\downarrow) \xrightarrow{(iii)} 0 \quad (\theta' \rightarrow \theta_0).
\end{aligned}$$

Here, (i) is due to mean value theorem, where $\bar{\theta}$ is a point between θ' and θ_0 . (ii) is due to

$|\mathbb{E}_{\theta'} [T_n^j | \mathfrak{B}_{r,\kappa}^\downarrow] - \mathbb{E}_{\theta_0} [T_n^j | \mathfrak{B}_{r,\kappa}^\downarrow]| \leq 2a_{4,n}$ and Lemma 4. (iii) is due to Lemma 10. Therefore, the second term equals

$$\lim_{\theta' \rightarrow \theta_0} \frac{\mathbb{E}_{\theta_0} [\mathbb{E}_{\theta'} [T_n^j | \mathfrak{B}_{r,\kappa}^\downarrow] - \mathbb{E}_{\theta_0} [T_n^j | \mathfrak{B}_{r,\kappa}^\downarrow]]}{\|\theta' - \theta_0\|} = \partial_i \mathbb{E}_{\theta_0} \mathbb{E}_{\theta} [T_n^j | \mathfrak{B}_{r,\kappa}^\downarrow]_{\theta=\theta_0}. \text{ After all, we have (15). } \blacksquare$$

4.4 ON LOGARITHMIC DERIVATIVE

Applying Leibniz rule to the indicator function, we can prove that $\left[\partial_i \int_{\Delta} P_{\theta}^{\mathbf{M}_{r,\kappa+1}^{\uparrow, \mathbf{z}_r}} (\Delta') dP_{\theta_0}^{\mathbf{M}_{r,\kappa}^{\downarrow}} (z_{r,\kappa}) \right]_{\theta=\theta_0}$

is finite. However, in general, one cannot prove existence of $\partial_i P_{\theta}^{\mathbf{M}_{r,\kappa+1}^{\uparrow, \mathbf{z}_r}} (\Delta')$. Therefore, we cannot define logarithmic derivative of the conditional probability distribution $P_{\theta}^{\mathbf{M}_{r,\kappa+1}^{\uparrow, \mathbf{z}_r}}$, nor cannot use the argument in Hayashi and Matsumoto (1998) in semi-classical case.

In one-way semi-classical case, which is treated in Hayashi and Matsumoto (1998), one can safely define the logarithmic derivative of $P_{\theta}^{\mathbf{M}_{r,\kappa+1}^{\uparrow, \mathbf{z}_r}}$, since $P_{\theta}^{\mathbf{M}_{r+1, r+1}^{\uparrow, \mathbf{z}_r}} (dz_{r+1, r+1}) = \operatorname{tr} \rho_{\theta} M_{r, \kappa+1}^{\uparrow, \mathbf{z}_r} (dz_{r+1, r+1})$. Therefore, their argument can be made rigorous, though we do not go into detail.

4.5 PROOF OF THEOREM 9

Observe that the estimator (10) with $n_1 = 1$ is one-way semi-classical. Therefore, the achievability by (one-way) semi-classical measurement follows from Lemmas 5-6. Therefore, below we prove the lowerbound.

In case , $\rho_{\theta} > 0$ ($\forall \theta \in \Theta$), due to the proof of lowerbound part of Theorem 1, we have the following lowerbound.

$$C_{\theta} (G_{\theta}, \mathcal{M}) \geq \overline{\lim}_{n \rightarrow \infty} \inf \{ n \operatorname{Tr} G_{\theta} V_{\theta} [\mathcal{E}_{\theta, n}] ; \text{ semi-classical, (2), (E')} \}$$

In the following, we reduce the optimization over semi-classical measurements to the one over *independent semi-classical measurements*, or one-way semi-classical measurements such that $\mathbf{N}_{\theta_0, \kappa}$ acting on κ -th sample cannot depend on the data $y_{\kappa'}$ from $\mathbf{N}_{\theta_0, \kappa'}$ ($\kappa \neq \kappa'$).

Lemma 12 Suppose that semi-classical estimator $\mathcal{E}_{\theta_0,n} = \{T_{\theta_0,n}, \mathbf{M}_{\theta_0}^n\}$ satisfies (2), (E'). Suppose also $\rho_\theta > 0, \forall \theta \in \Theta$. Then, we can find an estimator $\mathcal{F}_{n,\theta_0} = \{S_{\theta_0,n}, \mathbf{N}_{\theta_0}^n\}$, such that $\mathbf{N}_{\theta_0}^n$ is independent semi-classical, $V_{\theta_0}[\mathcal{F}_{\theta_0,n}] \leq V_{\theta_0}[\mathcal{E}_{\theta_0,n}]$, (2), and (E') hold. Moreover, $S_{\theta_0,n}$ is in the form of (17), where $F_{\theta_0,\kappa}$ is the function of the data y_κ from $\mathbf{N}_{\theta_0,\kappa}$, such that $E_{\theta_0}[F_{\theta_0,\kappa}] = 0$:

$$S_{\theta_0,n} = \sum_{\kappa=1}^n F_{\theta_0,\kappa}(y_\kappa) + \theta_0. \quad (17)$$

Proof. Since $E_\theta[T_{\theta,n}|\mathfrak{B}_{r,\kappa+1}^\downarrow]$ satisfies (E'), we apply Leibniz rule (15) recursively to obtain

$$\begin{aligned} \partial_i E_\theta [T_{\theta_0,n}]_{\theta=\theta_0} &= \frac{\partial}{\partial \theta^i} \left(E_{\theta_0} \left[E_\theta \left[T_{\theta_0,n} | \mathfrak{B}_{R_n,n-1}^\downarrow \right] \right] \right)_{\theta=\theta_0} + \frac{\partial}{\partial \theta^i} \left(E_\theta \left[E_{\theta_0} \left[T_{\theta_0,n} | \mathfrak{B}_{R_n,n-1}^\downarrow \right] \right] \right)_{\theta=\theta_0} \\ &= \frac{\partial}{\partial \theta^i} \left(E_{\theta_0} \left[E_\theta \left[T_{\theta_0,n} | \mathfrak{B}_{R_n,n-1}^\downarrow \right] \right] \right)_{\theta=\theta_0} + \frac{\partial}{\partial \theta^i} \left(E_{\theta_0} \left[E_\theta \left[E_{\theta_0} \left[T_{\theta_0,n} | \mathfrak{B}_{R_n,n-1}^\downarrow \right] | \mathfrak{B}_{R_n,n-2}^\downarrow \right] \right] \right)_{\theta=\theta_0} \\ &\quad + \frac{\partial}{\partial \theta^i} \left(E_\theta \left[E_{\theta_0} \left[E_{\theta_0} \left[T_{\theta_0,n} | \mathfrak{B}_{R_n,n-1}^\downarrow \right] | \mathfrak{B}_{R_n,n-2}^\downarrow \right] \right] \right)_{\theta=\theta_0} \\ &= \sum_{r=1}^{R_n} \sum_{\kappa=1}^n \frac{\partial}{\partial \theta^i} \left(E_{\theta_0} \left[E_{\theta_0} \left[\cdots E_\theta \left[E_{\theta_0} \left[\cdots E_{\theta_0} \left[T_{\theta_0,n} | \mathfrak{B}_{R_n,n-1}^\downarrow \right] \cdots | \mathfrak{B}_{r,\kappa+1}^\downarrow \right] | \mathfrak{B}_{r,\kappa}^\downarrow \right] \cdots | \mathfrak{B}_{1,1}^\downarrow \right] \right] \right)_{\theta=\theta_0} \\ &= \sum_{r=1}^{R_n} \sum_{\kappa=1}^n \frac{\partial}{\partial \theta^i} \left(E_{\theta_0} \left[E_\theta \left[E_{\theta_0} \left[T_{\theta_0,n} | \mathfrak{B}_{r,\kappa+1}^\downarrow \right] | \mathfrak{B}_{r,\kappa}^\downarrow \right] \right] \right)_{\theta=\theta_0}. \end{aligned}$$

Observe that, conditioned by \mathfrak{B}_{r-1} , the random variable $Z_{r,\kappa}$ and $Z_{r,\kappa'}$ are independent, due to the composition of the measurement. Therefore, due to Fubini's theorem,

$$\begin{aligned} E_{\theta_0} \left[E_\theta \left[E_{\theta_0} \left[T_{\theta_0,n} | \mathfrak{B}_{r,\kappa+1}^\downarrow \right] | \mathfrak{B}_{r,\kappa}^\downarrow \right] | \mathfrak{B}_{r-1} \right] &= \int E_{\theta_0} [T_{\theta_0,n} | \mathfrak{B}_r] \prod_{\kappa': \kappa' \neq \kappa} dP_{\theta_0}(z_{r,\kappa'} | \mathfrak{B}_{r-1}) dP_\theta(z_{r,\kappa} | \mathfrak{B}_{r-1}) \\ &= E_\theta [E_{\theta_0} [T_{\theta_0,n} | \langle \mathfrak{B}_{r-1}, \mathfrak{B}_{r,\kappa} \rangle] | \mathfrak{B}_{r-1}]. \end{aligned}$$

Therefore,

$$\partial_i E_\theta [T_{\theta_0,n}]_{\theta=\theta_0} = \sum_{r=1}^{R_n} \sum_{\kappa=1}^n \frac{\partial}{\partial \theta^i} (E_{\theta_0} [E_\theta [E_{\theta_0} [T_{\theta_0,n} | \langle \mathfrak{B}_{r-1}, \mathfrak{B}_{r,\kappa} \rangle] | \mathfrak{B}_{r-1}]]_{\theta=\theta_0}). \quad (18)$$

Let us define, with the convention $\mathfrak{B}_0 = \{\emptyset, \mathbb{R}^l\}$,

$$f_{\theta_0,r,\kappa} := E_{\theta_0} [T_{\theta_0,n} | \langle \mathfrak{B}_{r-1}, \mathfrak{B}_{r,\kappa} \rangle] - E_{\theta_0} [T_n | \mathfrak{B}_{r-1}], \quad F_{\theta_0,\kappa} := \sum_{r=1}^{R_n} f_{\theta_0,r,\kappa}.$$

Since $f_{\theta_0,r,\kappa}$ also satisfies (E'), we can apply Leibniz rule (15). Therefore,

$$\begin{aligned} \partial_i E_\theta f_{\theta_0,r,\kappa} |_{\theta=\theta_0} &= \partial_i E_\theta [E_\theta [f_{\theta_0,r,\kappa} | \mathfrak{B}_{r-1}]] |_{\theta=\theta_0} \\ &= (\partial_i E_{\theta_0} [E_\theta [f_{\theta_0,r,\kappa} | \mathfrak{B}_{r-1}]] + \partial_i E_\theta [E_{\theta_0} [f_{\theta_0,r,\kappa} | \mathfrak{B}_{r-1}]])_{\theta=\theta_0} \stackrel{(i)}{=} \partial_i E_{\theta_0} [E_\theta [f_{\theta_0,r,\kappa} | \mathfrak{B}_{r-1}]] |_{\theta=\theta_0} \\ &= \partial_i (E_{\theta_0} [E_\theta [(E_{\theta_0} [T_{\theta_0,n} | \langle \mathfrak{B}_{r-1}, \mathfrak{B}_{r,\kappa} \rangle] - E_{\theta_0} [T_{\theta_0,n} | \mathfrak{B}_{r-1}]) | \mathfrak{B}_{r-1}]])_{\theta=\theta_0} \\ &\stackrel{(ii)}{=} \partial_i E_{\theta_0} [E_\theta [E_{\theta_0} [T_{\theta_0,n} | \langle \mathfrak{B}_{r-1}, \mathfrak{B}_{r,\kappa} \rangle] | \mathfrak{B}_{r-1}]] |_{\theta=\theta_0}. \end{aligned} \quad (19)$$

Here, (i) is due to $E_{\theta_0} [f_{\theta_0,r,\kappa} | \mathfrak{B}_{r-1}] = E_{\theta_0} [(E_{\theta_0} [T_{\theta_0,n} | \langle \mathfrak{B}_{r-1}, \mathfrak{B}_{r,\kappa} \rangle] - E_{\theta_0} [T_n | \mathfrak{B}_{r-1}]) | \mathfrak{B}_{r-1}] = 0$, and

(ii) is due to $E_\theta [E_{\theta_0} [T_{\theta_0,n} | \mathfrak{B}_{r-1}] | \mathfrak{B}_{r-1}] = E_{\theta_0} [T_{\theta_0,n} | \mathfrak{B}_{r-1}]$. Combining (18) and (19), we have $\partial_i E_\theta [T_{\theta_0,n}]_{\theta=\theta_0} =$

$\sum_{r=1}^{R_n} \sum_{\kappa=1}^n \partial_i \mathbb{E}_\theta [f_{\theta_0, r, \kappa}]|_{\theta=\theta_0}$. Therefore, with $S'_{\theta_0, n} := \sum_{\kappa=1}^n F_{\theta_0, \kappa}(\mathbf{Z}_{R_n}) + \theta_0$, $\{\mathbf{M}_{\theta_0}^n, S'_{\theta_0, n}\}$ is locally unbiased at θ_0 . Also, observe the following relations:

$$\mathbb{E}_{\theta_0} f_{\theta_0, r, \kappa} (f_{\theta_0, r', \kappa'})^T = 0 \quad (\kappa \neq \kappa' \text{ or } r \neq r'), \quad \mathbb{E}_{\theta_0} f_{\theta_0, r, \kappa} (T_{\theta_0, n})^T = \mathbb{E}_{\theta_0} f_{\theta_0, r, \kappa} (f_{\theta_0, r, \kappa})^T.$$

Due to them, the variance of this estimate is not larger than the one of $T_{\theta_0, n}$:

$$\mathbb{V}_{\theta_0} \left[\left\{ \mathbf{M}_{\theta_0}^n, S'_{\theta_0, n} \right\} \right] = \sum_{\kappa=1}^n \sum_{r=1}^{R_n} \mathbb{E}_{\theta_0} f_{\theta_0, r, \kappa} (f_{\theta_0, r, \kappa})^T = \sum_{\kappa=1}^n \sum_{r=1}^{R_n} \mathbb{E}_{\theta_0} \left[f_{\theta_0, r, \kappa} (T_{\theta_0, n})^T \right] \leq \mathbb{V}_{\theta_0} [\mathcal{E}_{\theta_0, n}].$$

Below, we define $\mathbf{N}_{\theta_0, \kappa}^n$. First, using $\rho_{\theta_0}^{\otimes n}$, we prepare n of *fake ensembles* $\rho_{\theta_0} \otimes \rho_{\theta_0} \cdots \otimes \rho_{\theta_0} \otimes \cdots \otimes \rho_{\theta_0}$ ($\kappa = 1, \dots, n$), composed with single ρ_{θ} and $n-1$ of ρ_{θ_0} . Then $\mathbf{N}_{\theta_0, \kappa}^n$ is the application of $\mathbf{M}_{\theta_0}^{\kappa}$ to κ -th fake ensemble. We denote by $z_{r, \kappa}^{(\kappa)}$ the data obtained at r -th round from the κ -th (possibly fake) sample in the κ -th fake ensemble. The data y_{κ} from $\mathbf{N}_{\theta_0, \kappa}^n$ is $y_{\kappa} := \mathbf{z}_{R_n}^{(\kappa)}$.

If $\theta = \theta_0$, $Y_{\kappa} = \mathbf{Z}_{R_n}^{(\kappa)}$ obeys the same probability distribution as \mathbf{Z}_{R_n} , for any κ . Therefore, $\mathbb{V}_{\theta_0} [\{\mathbf{N}_{\theta_0}^n, F_{\theta_0, \kappa}(Y_{\kappa})\}]$ equals $\mathbb{V}_{\theta_0} [\{\mathbf{M}_{\theta_0}^n, F_{\theta_0, \kappa}(\mathbf{Z}_{R_n})\}]$. Therefore, due to $\mathbb{V}_{\theta_0} [\{\mathbf{M}_{\theta_0}^n, S'_{\theta_0, n}\}] = \sum_{\kappa=1}^n \mathbb{V}_{\theta_0} [F_{\theta_0, \kappa}]$, we have $\mathbb{V}_{\theta_0} [\mathcal{F}_{n, \theta_0}] = \mathbb{V}_{\theta_0} [\{\mathbf{M}_{\theta_0}^n, S'_{\theta_0, n}\}] \leq \mathbb{V}_{\theta_0} [\mathcal{E}_{\theta_0, n}]$. Analogously, we can also show $\mathbb{E}_{\theta_0}^{\mathbf{N}_{\theta_0}^n} [S_{\theta_0, n}] = \mathbb{E}_{\theta_0} [S'_{\theta_0, n}] = \theta_0$.

Finally, we show $\partial_i \mathbb{E}_{\theta}^{\mathbf{N}_{\theta_0}^n} [S_{\theta_0, n}^j] \Big|_{\theta=\theta_0} = \partial_i \mathbb{E}_{\theta} [S'_{\theta_0, n}^j] \Big|_{\theta=\theta_0} = \delta_i^j$. Observe

$$\begin{aligned} & \frac{\partial}{\partial \theta^i} \left(\mathbb{E}_{\theta}^{\mathbf{N}_{\theta_0}^n} f_{\theta_0, \kappa, r} \left(\mathbf{z}_{r-1}^{(\kappa)}, z_{r, \kappa}^{(\kappa)} \right) \right)_{\theta=\theta_0} \\ &= \frac{\partial}{\partial \theta^i} \left[\int f_{\theta_0, \kappa, r} \left(\mathbf{z}_{r-1}^{(\kappa)}, z_{r, \kappa}^{(\kappa)} \right) \prod_{r'=1}^r \left(\prod_{\kappa': \kappa' \neq \kappa} dP_{\theta_0} \left(z_{r', \kappa'}^{(\kappa)} | \mathbf{z}_{r'-1}^{(\kappa)} \right) dP_{\theta} \left(z_{r', \kappa}^{(\kappa)} | \mathbf{z}_{r'-1}^{(\kappa)} \right) \right) \right]_{\theta=\theta_0} \\ &= \partial_i (\mathbb{E}_{\theta} [\mathbb{E}_{\theta_0} [\cdots \mathbb{E}_{\theta} [\mathbb{E}_{\theta_0} [\mathbb{E}_{\theta} [f_{\theta_0, \kappa, r} | \mathfrak{B}_{r-1}] | \langle \mathfrak{B}_{r-1, \kappa}, \mathfrak{B}_{r-2} \rangle] | \mathfrak{B}_{r-2}] \cdots | \mathfrak{B}_{1, \kappa}]]])_{\theta=\theta_0} \\ &= \partial_i (\mathbb{E}_{\theta_0} [\mathbb{E}_{\theta} [f_{\theta_0, \kappa, r} | \mathfrak{B}_{r-1}]]])_{\theta=\theta_0} + \sum_{r'=3}^{r-1} \partial_i (\mathbb{E}_{\theta_0} [\mathbb{E}_{\theta} [\mathbb{E}_{\theta_0} [f_{\theta_0, \kappa, r} | \langle \mathfrak{B}_{r', \kappa}, \mathfrak{B}_{r'-1} \rangle] | \mathfrak{B}_{r'-2}]]])_{\theta=\theta_0} \\ &\quad + \partial_i (\mathbb{E}_{\theta} [\mathbb{E}_{\theta_0} [f_{\theta_0, \kappa, r} | \mathfrak{B}_{1, \kappa}]]])_{\theta=\theta_0} \end{aligned}$$

where the last equality is due to Leibniz rule. Due to the definition of $f_{\theta_0, \kappa, r}$, $\mathbb{E}_{\theta_0} [f_{\theta_0, \kappa, r} | \langle \mathfrak{B}_{r', \kappa}, \mathfrak{B}_{r'-1} \rangle]$ ($r' \leq r-1$) and $\mathbb{E}_{\theta_0} [f_{\theta_0, \kappa, r} | \mathfrak{B}_{1, \kappa}]$ are zero. Therefore, $\partial_i \mathbb{E}_{\theta}^{\mathbf{N}_{\theta_0}^n} [S_{\theta_0, n}^j] \Big|_{\theta=\theta_0} = \partial_i \mathbb{E}_{\theta} [S'_{\theta_0, n}^j] \Big|_{\theta=\theta_0} = \delta_i^j$ follows from (19). Trivially, $\mathcal{F}_{n, \theta_0} = \{\mathbf{N}_{\theta_0}^n, S_{\theta_0, n}\}$ satisfies (E'). After all, we have the lemma. ■

Lemma 13 Suppose $\mathbf{N}_{\theta_0}^n$ is independent semi-classical. Suppose also that $S_{\theta_0, n}$ is in the form of (17), and that $\mathcal{F}_{n, \theta_0} = \{\mathbf{N}_{\theta_0}^n, S_{\theta_0, n}\}$ satisfies (2) and (E'). Then, we can find an estimator $\mathcal{E}'_{\theta_0, 1} = \{\mathbf{M}'_{\theta_0}, T'_{\theta_0, 1}\}$ acting on single sample, with (2), (E'), and $\mathbb{V}_{\theta_0} [\mathcal{E}'_{\theta_0, 1}] = n \mathbb{V}_{\theta_0} [\mathcal{F}_{n, \theta_0}]$.

Proof. \mathbf{M}'_{θ_0} is constructed as follows; generate $x_{\kappa} \in \{1, \dots, n\}$ according to uniform distribution, and apply $\mathbf{N}_{\theta_0, x_{\kappa}}$ to ρ_{θ} , generating the data y_{κ} . The data resulting from \mathbf{M}'_{θ_0} is the pair $y'_{\kappa} := (x_{\kappa}, y_{x_{\kappa}})$. $T'_{\theta_0, 1}$ is defined by $T'_{\theta_0, 1}(y'_{\kappa}) := n F_{\theta_0, x_{\kappa}}(y_{x_{\kappa}}) + \theta_0$.

Observe $\mathbb{E}_{\theta}^{\mathbf{M}'_{\theta_0}} [T'_{\theta_0, 1}] = \frac{1}{n} \sum_{\kappa=1}^n n \mathbb{E}_{\theta}^{\mathbf{N}_{\theta_0, \kappa}} [F_{\theta_0, \kappa}] + \theta_0 = \mathbb{E}_{\theta}^{\mathbf{N}_{\theta_0}^n} [S_{\theta_0, n}]$, implying (2) for \mathcal{E}'_{θ_0} . MSE of \mathcal{E}'_{θ_0} is computed as follows: $\mathbb{V}_{\theta_0} [\mathcal{E}'_{\theta_0, 1}] = \frac{1}{n} \sum_{\kappa=1}^n n^2 \mathbb{E}_{\theta}^{\mathbf{N}_{\theta_0}^n} [F_{\theta_0, \kappa}(Y_{\kappa}) F_{\theta_0, \kappa}(Y_{\kappa})^T] = n \mathbb{V}_{\theta_0} [\mathcal{F}_{n, \theta_0}]$. (2) and (E') for \mathcal{E}'_{θ_0} are trivial. ■

Due to Lemma 10, the above two lemmas leads to ‘ \geq ’- part of the first identity of Theorem 9, in the case where $\rho_\theta > 0$, $\forall \theta \in \Theta$. The statement for the general case is straightforward consequence of the following lemma.

Lemma 14 *Let $\mathcal{E}_{\theta,n} := \{\mathbf{M}_\theta^n, T_{\theta,n}\}$ be a locally unbiased estimator at θ with (E’). Then, $\exists V \geq 0$, $\forall \varepsilon > 0$, $\exists \mathcal{E}'_{\varepsilon,\theta,1} := \{\mathbf{M}'_\theta, T'_{\varepsilon,\theta,1}\}$ acting on a single sample with (E’) and $V_\theta[\mathcal{E}_{\theta,n}] \geq \frac{1}{n} V_\theta[\mathcal{E}'_{\theta,1}] - \frac{\varepsilon}{(1-\varepsilon)} V$.*

Proof. Let $\sigma > 0$, and define $\rho_{\theta,\varepsilon} := (1-\varepsilon)\rho_\theta + \varepsilon\sigma$. Denote by $E_{\theta,\varepsilon}[T_{\theta,n}]$ and $V_{\theta,\varepsilon}[\mathcal{E}_{\theta,n}]$ the average and the variance of $\mathcal{E}_{\theta,n}$ with respect to $\rho_{\theta,\varepsilon}$. Then, there is a locally unbiased estimator $\mathcal{E}_{\theta,\varepsilon,n} := \{\mathbf{M}_\theta^n, T_{\theta,\varepsilon,n}\}$ with respect to the quantum statistical model $\{\rho_{\theta,\varepsilon}\}_{\theta \in \Theta}$ which satisfies (E) and the relation $T_{\theta,\varepsilon,n} = \frac{1}{1-\varepsilon}(T_{\theta,n} - E_{\theta,\varepsilon}[T_{\theta,n}]) + \theta$. Since $\rho_{\theta,\varepsilon} > 0$ and the family $\{\rho_{\theta,\varepsilon}\}_{\theta \in \Theta}$ satisfies (M.1,2), Lemmas 12-13 imply existence of a locally unbiased estimator $\mathcal{E}'_{\varepsilon,\theta,1} := \{\mathbf{M}'_\theta, T'_{\varepsilon,\theta,1}\}$ acting on single sample such that (E’) and $V_{\theta,\varepsilon}[\mathcal{E}_{\theta,\varepsilon,n}] \geq \frac{1}{n} V_{\theta,\varepsilon}[\mathcal{E}'_{\theta,\varepsilon,1}]$.

Defining $\mathcal{E}'_{\theta,1} := \{\mathbf{M}'_\theta, T'_{\theta,1}\}$ by $T'_{\theta,1} := (1-\varepsilon)(T'_{\varepsilon,\theta,1} - E_\theta[T'_{\varepsilon,\theta,1}]) + \theta$, $\mathcal{E}'_{\theta,1}$ is locally unbiased with respect to $\{\rho_\theta\}_{\theta \in \Theta}$. Letting V be the variance of $\mathcal{E}_{\theta,n}$ with respect to σ , we have $V_\theta[\mathcal{E}_{\theta,n}] = (1-\varepsilon)V_{\theta,\varepsilon}[\mathcal{E}_{\theta,\varepsilon,n}] - \frac{\varepsilon}{1-\varepsilon}V \geq \frac{1-\varepsilon}{n}V_{\theta,\varepsilon}[\mathcal{E}'_{\theta,\varepsilon,1}] - \frac{\varepsilon}{1-\varepsilon}V \geq \frac{1}{n}V_\theta[\mathcal{E}'_{\theta,1}] - \frac{\varepsilon}{1-\varepsilon}V$. ■

Combining with the achievability, the first identity of Theorem 9 is proved. The second identity is shown using the analogous argument as the proof of (3)=(4), and Theorem 9 is proved.

Note that the estimator achieving the lowerbound is one-way semi-classical. This means that the optimal asymptotic cost of one-way semi-classical measurements is also $C_\theta(G_\theta, \mathcal{M})$.

5. LOCC STATE ESTIMATION

Suppose the state $\rho_\theta^{\otimes n}$ is shared by remote party, Alice and Bob: $\mathcal{H} = \mathcal{H}_a \otimes \mathcal{H}_b$. Suppose also that Alice and Bob can exchange classical messages, but cannot interact quantumly with each other. So they do the following R_n -round measurement \mathbf{M}^n : At each round, Alice and Bob measures her/his share of the samples, and the measurements of the r -th round depend on the previously obtained data. We denote by $\xi_r^a (\in \mathbb{R}^l)$ and $\xi_r^b (\in \mathbb{R}^l)$ the data obtained at the r -th round by Alice’s and Bob’s measurement, respectively. Also, $\vec{\xi}_r$ denotes $(\xi_r^a, \xi_r^b)_{r'=1}^r$. We denote by \mathfrak{C}_r and \mathfrak{C}_r^x ($x = a, b$) the Borel field over the space which $\vec{\xi}_{r-1}$ and ξ_r^x takes values in, respectively. The measurement acting in the r -th round by Alice (Bob) is denoted by $\mathbf{M}_{r,a}^{\vec{\xi}_{r-1}}$ ($\mathbf{M}_{r,b}^{\vec{\xi}_{r-1}}$, resp.). Such operations are said to be *local operations and quantum communications (LOCC*, in short). An important subclass of LOCC is *local operation (LO)*, where Alice and Bob does \mathbf{M}_a^n and \mathbf{M}_b^n , independently.

The difference between semi-classical and LOCC measurements is the split of the actions. In the former, split is between samples. In the latter, the split is between Alice and Bob. Other than this point, basically they are the same concept. Especially, LO corresponds to independent semi-classical measurements.

Define $C_\theta^{Q,L}(G_\theta, \mathcal{M})$ and $C_\theta^L(G_\theta, \mathcal{M})$ by restricting the range of measurement to LOCC in the definition of $C_\theta^Q(G_\theta, \mathcal{M})$ and $C_\theta(G_\theta, \mathcal{M})$, respectively. Then, trivially, we have:

Theorem 15 *Suppose (M.1-3) hold. Then,*

$$C_\theta^L(G_\theta, \mathcal{M}) = \inf \{ \text{Tr } G_{\theta_0} V_{\theta_0} [\mathcal{E}_{\theta_0,1}] ; \text{ LOCC, (2), (E) for } n=1 \},$$

$$C_\theta^{Q,L}(G_\theta, \mathcal{M}) = \lim_{n \rightarrow \infty} \inf \{ n \text{Tr } G_{\theta_0} V_{\theta_0} [\mathcal{E}_{\theta_0,n}] ; \text{ LOCC, (2), (E) } \}.$$

From here, we focus on the case of $\rho_\theta = \rho_\theta^a \otimes \rho_\theta^b$. The motivation of studying this seemingly easy case is as follows. Suppose that $\text{rank} \rho_\theta = 1$, $\dim \mathcal{H} < \infty$, and $\rho_\theta \neq \rho_\theta^a \otimes \rho_\theta^b$. Then it is known that $C_\theta^L = C_\theta^{Q,L} = C_\theta = C_\theta^Q$ (Matsumoto (2007)). The estimator used to show the identity, however, fails in case of $\rho_\theta = \rho_\theta^a \otimes \rho_\theta^b$.

In this case, we can translate the argument in Section 4 about the lowerbound to the asymptotic cost of semi-classical estimators to LOCC estimators. To see this, observe the proof of Lemmas 11-12 are valid even if non-identical independent samples $\bigotimes_{\kappa=1}^n \rho_\theta^{(\kappa)}$ are given. Below, we present the analogue of Lemma 12. (The proof is omitted being almost parallel.) Using this lemma, the optimization over LOCC is reduced to the one over LO, where $\mathbf{N}_{\theta_0,n}^a$ and $\mathbf{N}_{\theta_0,n}^b$ is measured independently, producing the data y^a, y^b .

Lemma 16 *Suppose $\rho_\theta = \rho_\theta^a \otimes \rho_\theta^b > 0$, $\forall \theta \in \Theta$. Suppose an LOCC estimator $\mathcal{E}_{\theta_0,n} = \{\mathbf{M}_{\theta_0,n}^n, T_{\theta_0,n}\}$ satisfies (2) and (E'). Then, we can find an LO estimator $\mathcal{F}_{\theta_0,n} = \{\mathbf{N}_{\theta_0,n}, S_{\theta_0,n}\}$, such that $V_{\theta_0}[\mathcal{F}_{\theta_0,n}] \leq V_{\theta_0}[\mathcal{E}_{\theta_0,n}]$, (2), and (E') hold. Moreover, $S_{\theta_0,n}$ is in the following form:*

$$S_{\theta_0,n} = F_{\theta_0,n}^a(\xi_n^a) + F_{\theta_0,n}^b(\xi_n^b) + \theta_0. \quad (20)$$

Let \mathcal{M}_x denote $\{\rho_\theta^x\}_{\theta \in \Theta}$ ($x = a, b$). Observe that the map $\theta \rightarrow \rho_\theta^x$ may not be injective. Therefore, the vector space $\{\mathbf{v}_\theta^x; \sum_{i=1}^m v^{x,i} \partial_i \rho_\theta^x = 0\}$ may not be $\{0\}$. We denote by Π_θ^x the projector onto the orthogonal complement of this vector space in \mathbb{R}^m . Letting $\mathcal{F}_{\theta_0,n}^x := \{\mathbf{N}_{\theta_0,n}^x, F_{\theta_0,n}^x\}$, $B_{\theta_0}[\mathcal{F}_{\theta_0,n}^x] \mathbf{v}_{\theta_0}^x = 0$ if $\Pi_{\theta_0}^x \mathbf{v}_{\theta_0}^x = 0$. Therefore, there is a matrix $W_{\theta_0}^x$ with $B_{\theta_0}[\mathcal{F}_{\theta_0,n}^x] = W_{\theta_0}^x \Pi_{\theta_0}^x$.

We want to minimize the variance of locally unbiased estimator in the form of (20). First, for a given $(F_{\theta_0,n}^a, F_{\theta_0,n}^b)$, we define $S_{\theta_0,n}[A_{\theta_0,n}^a, A_{\theta_0,n}^b] := A_{\theta_0,n}^a F_{\theta_0,n}^a(\xi_n^a) + A_{\theta_0,n}^b F_{\theta_0,n}^b(\xi_n^b) + \theta_0$, where $(A_{\theta_0,n}^a, A_{\theta_0,n}^b)$ moves over all the $m \times m$ real invertible matrices. Elementary but tedious calculation shows that the variance of such estimators is larger (in the sense that the difference is positive semi-definite) than

$$\left(\Pi_{\theta_0}^a (V_{\theta_0}[\mathcal{E}_{\theta_0,n}^a])^{-1} \Pi_{\theta_0}^a + \Pi_{\theta_0}^b (V_{\theta_0}[\mathcal{E}_{\theta_0,n}^b])^{-1} \Pi_{\theta_0}^b \right)^{-1}, \quad (21)$$

where $\mathcal{E}_{\theta_0,n}^x := \{\mathbf{N}_{\theta_0,n}^x, \tilde{F}_{\theta_0,n}^x\}$, $\tilde{F}_{\theta_0,n}^x := (W_{\theta_0}^x)^{-1} F_{\theta_0,n}^x$ and $(\cdot)^{-1}$ in (21) denotes generalized inverse. Observe that $\mathcal{E}_{\theta_0,n}^x$ ($x = a, b$) satisfies

$$B_{\theta_0}[\mathcal{E}_{\theta_0,n}^x] = \Pi_{\theta_0}^x \quad (x = a, b). \quad (22)$$

(CM.1)	$\partial_i \Lambda_\theta, \partial_i \partial_j \Lambda_\theta$ exists and are locally uniformly continuous, $\ \partial_i \Lambda_\theta\ _{cb}, \ \partial_i \partial_j \Lambda_\theta\ _{cb} \leq \frac{a_1}{2} < \infty, \forall \theta \in \Theta$
(CE)	$\int \ T_n(\omega) - \theta\ \text{tr} \varpi_1^n(\square_{i,\theta}, E_n) M^n(d\omega_n) \leq n a_1 a_{4,n}, \int \ T_n(\omega) - \theta\ ^2 \text{tr} P_\theta^{E_n}(d\omega_n) \rho_\theta^{\otimes n} \leq n (a_{4,n})^2, \forall \theta \in \Theta$

Table 2: Regularity conditions on models and estimators in operation estimation.

In the end, we optimize (21) with the constrain (2) and (E'). Here, constrain (E') can be replaced by (E) without increasing the infimum, due to the analogous argument as the proof of (3)=(4).

So far we had assumed $\rho_\theta > 0$. However, this assumption can be removed due to an analogue of Lemma 14 (the proof is omitted, being straightforward.). Therefore, letting $\mathcal{V}_{\theta,n}^x(\mathcal{M})$ the totality of $V_\theta[\mathcal{E}_{\theta,n}]$ with (22) and (E) ($x = \mathbf{a}, \mathbf{b}$), we have

Theorem 17 Suppose $\rho_\theta = \rho_\theta^{\mathbf{a}} \otimes \rho_\theta^{\mathbf{b}}, \forall \theta \in \Theta$. Then,

$$C_\theta^L(G_\theta, \mathcal{M}) \geq \inf \left\{ \text{Tr} G_\theta (\Pi_\theta^{\mathbf{a}} V_{\mathbf{a}}^{-1} \Pi_\theta^{\mathbf{a}} + \Pi_\theta^{\mathbf{b}} V_{\mathbf{b}}^{-1} \Pi_\theta^{\mathbf{b}})^{-1}, V_x \in \mathcal{V}_{\theta,1}^x(\mathcal{M}), x = \mathbf{a}, \mathbf{b} \right\},$$

$$C_\theta^{Q,L}(G_\theta, \mathcal{M}) \geq \lim_{n \rightarrow \infty} \inf \left\{ n \text{Tr} G_\theta (\Pi_\theta^{\mathbf{a}} V_{\mathbf{a}}^{-1} \Pi_\theta^{\mathbf{a}} + \Pi_\theta^{\mathbf{b}} V_{\mathbf{b}}^{-1} \Pi_\theta^{\mathbf{b}})^{-1}, V_x \in \mathcal{V}_{\theta,n}^x(\mathcal{M}), x = \mathbf{a}, \mathbf{b} \right\}.$$

The achievability of the lowerbound is also true. This theorem leads to a necessary and sufficient condition for $C_\theta^L(G_\theta, \mathcal{M})$ ($C_\theta(G_\theta, \mathcal{M})$) to equal $C_\theta^{Q,L}(G_\theta, \mathcal{M})$ ($C_\theta^Q(G_\theta, \mathcal{M})$, resp.). These topics, however, will be discussed elsewhere.

6. ESTIMATION OF QUANTUM OPERATIONS

Suppose we are given a family of completely positive and trace preserving maps $\mathcal{L} := \{\Lambda_\theta\}_{\theta \in \Theta}$. Here, $\Lambda_\theta : \mathcal{B}(\mathcal{H}) \rightarrow \mathcal{B}(\mathcal{H}')$, $\theta \in \Theta$, and Θ is an open region in \mathbb{R}^m . Our purpose is to estimate θ , by measuring the output of Λ_θ after sending the input state for n times through it.

Our input state ρ^n is living in $\mathcal{H} \otimes \mathcal{K}$, where $\dim \mathcal{K}$ is arbitrarily large. \mathcal{K} may be used to store the input state before and/or after application of Λ_θ . Between the κ -th and $(\kappa + 1)$ -th use of Λ_θ , one can apply an operation $\Xi_\kappa^n : \mathcal{B}(\mathcal{H}' \otimes \mathcal{K}) \rightarrow \mathcal{B}(\mathcal{H} \otimes \mathcal{K})$. Ξ_κ^n may be a composition of measurement followed by preparation of the state to be send through Λ_θ . After n times of use of Λ_θ , we obtain $\prod_{\kappa=1}^n \{(\Lambda_\theta \otimes \mathbf{I}) \circ \Xi_\kappa^n\}(\rho^n)$. We measure this by M^n , obtaining the data $\omega_n \in \mathbb{R}^{l_n}$, and compute the estimate $T_n(\omega_n)$. The pair $\mathcal{E}_n := \{\rho^n, \{\Xi_\kappa^n\}_{\kappa=1}^{n-1}, M^n, T_n\}$ (, or sometimes sequence $\{\mathcal{E}_n\}_{n=1}^\infty$ also,) is called an *estimator*. The probability distribution of the data is $P_\theta^{E_n}\{\omega_n \in \Delta\} = \text{tr} M^n(\Delta) \prod_{\kappa=0}^n \{(\Lambda_\theta \otimes \mathbf{I}) \circ \Xi_\kappa^n\}(\rho^n)$.

Regularity conditions, other than (1) on estimators, are listed in Table 2. Note that they are honest analogue of (M.1) and (E). In the table, convergence is always in terms of $\|\cdot\|_{cb}$, and $\square_{i,\theta} : \mathcal{B}(\mathcal{H}) \rightarrow \mathcal{B}(\mathcal{H})$ is an affine map with $\square_{i,\theta} \otimes \mathbf{I}(\rho) \geq \partial_i \Lambda_{\theta_0} \otimes \mathbf{I}(\rho)$ ($\|\theta_0 - \theta\| < a_2$), whose existence is certified by (CM.1) and an analogue of Lemma 3. Also, $\varpi_1^n(\square_{i,\theta}, E_n)$ is defined by replacing $\partial_i \Lambda_\theta$ in $\partial_i \prod_{\kappa=0}^n \{(\Lambda_\theta \otimes \mathbf{I}) \circ \Xi_\kappa^n\}(\rho^n)$ by $\square_{i,\theta}$.

Define $C_\theta^{Q,Op}(G_\theta, \mathcal{L})$ by replacing (E) in the definition of $C_\theta^{Q,Op}(G_\theta, \mathcal{M})$ by (CE). Then we have, honestly modifying the argument in Section 3 (the proof is omitted),

Theorem 18 *If (CM.1) holds, $C_\theta^{Q,Op}(G_\theta, \mathcal{L}) \geq \liminf_{n \rightarrow \infty} \{n \text{Tr } G_\theta V_\theta [\mathcal{E}_{\theta,n}] ; (2), (E')\}$*

Also, the achievability of the lowerbound can be proved, with some additional regularity conditions. It is known that in case of $\Lambda_\theta(\rho) = U_\theta \rho U_\theta^\dagger$, with $U_\theta U_\theta^\dagger = U_\theta^\dagger U_\theta = \mathbf{1}$, there is an asymptotically unbiased estimator with $\text{Tr } G_\theta V_\theta [\mathcal{E}_n] = O(\frac{1}{n^\tau})$. Therefore, $C_\theta^{Q,C}(G_\theta, \mathcal{L}) = 0$ for such models. In this subsection, we show that such a phenomena can occur only at the surface of the space of the quantum operations.

Theorem 19 *Suppose (CM.1-2) and (CE) holds. Moreover, suppose $\exists \varepsilon > 0$ s.t., $\Lambda_\theta + \sum_{i=1}^m u^i \partial_i \Lambda_\theta$ is completely positive, for $\forall u : \|u\| < \varepsilon, \forall \theta \in \Theta$. Then, $C_\theta^{Q,C}(G_\theta, \mathcal{L}) \neq 0$.*

Proof. Define $\Lambda_{\theta_0, \theta} := \Lambda_{\theta_0} + \sum_{i=1}^m \frac{\partial \Lambda_{\theta_0}}{\partial \theta^i} (\theta^i - \theta_0^i)$, $\mathcal{L}_{\theta_0} := \{\Lambda_{\theta_0, \theta}; \sum_{i=1}^m |\theta^i - \theta_0^i| < \varepsilon\}$. Suppose $\mathcal{E}_{\theta_0, n} := \{\rho_{\theta_0}^n, \{\Xi_{\kappa, \theta_0}^n\}_{\kappa=1}^n, M_{\theta_0}^n, T_{\theta_0, n}\}$ satisfies (CE) and (2) at $\theta = \theta_0$ as an estimator of $\mathcal{L} := \{\Lambda_\theta\}_{\theta \in \Theta}$.

$$\begin{aligned} \partial_i \left[\int T_{\theta_0, n}^j(\omega_n) \text{tr} \left[M_{\theta_0}^n(d\omega_n) \prod_{\kappa=0}^n \left\{ \left(\Lambda_{\theta_0} + \sum_{i=1}^m \frac{\partial \Lambda_{\theta_0}}{\partial \theta^i} (\theta^i - \theta_0^i) \otimes \mathbf{I} \right) \circ \Xi_{\kappa, \theta_0}^n \right\} \rho_{\theta_0}^n \right] \right]_{\theta=\theta_0} \\ = \partial_i \left[\int T_{\theta_0, n}^j(\omega_n) \text{tr} \left[M_{\theta_0}^n(d\omega_n) \prod_{\kappa=0}^n \left\{ (\Lambda_\theta \otimes \mathbf{I}) \circ \Xi_{\kappa, \theta_0}^n \right\} \rho_{\theta_0}^n \right] \right]_{\theta=\theta_0} = \delta_i^j. \end{aligned}$$

Here, the first identity holds since (CE) implies an analogue of Lemma 4. Therefore, since $\Lambda_{\theta_0, \theta_0} = \Lambda_{\theta_0}$, $\mathcal{E}_{\theta_0, n}$ satisfies (2) at $\theta = \theta_0$ as an estimator of \mathcal{L}_{θ_0} , also. Moreover, the variance of $\mathcal{E}_{\theta_0, n}$ as an estimator of \mathcal{L}_{θ_0} and \mathcal{L} coincide at $\theta = \theta_0$. Therefore, it suffices to show the statement for the quantum operation model \mathcal{L}_{θ_0} . From here, we follow the same line of argument as Hayashi (2003) and Zhengfeng Ji, et. al. (2006).

Let θ_x be the x -th extreme point of the convex region $\sum_{i=1}^m |\theta^i - \theta_0^i| \leq \varepsilon$. Then, there is $p_{\theta_0, \theta}(x)$ such that $\Lambda_{\theta_0, \theta} = \sum_{x=1}^{2^m} p_{\theta_0, \theta}(x) \Lambda_{\theta_0, \theta_x}$, $\sum_{x=1}^{2^m} p_{\theta_0, \theta}(x) = 1$, and $p_{\theta_0, \theta}(x)$ is linear in θ . Consider the family of multinomial probability distributions $\{p_{\theta_0, \theta}(\cdot)\}_{\theta \in \Theta}$. The key observation is that $\Lambda_{\theta_0, \theta}$ is equivalent to random application of $\Lambda_{\theta_0, \theta_x}$, where x is sampled according to $p_{\theta_0, \theta}(\cdot)$. Following this observation, given a locally unbiased estimator $\mathcal{E}_{\theta_0, n}$ of \mathcal{L}_{θ_0} at $\theta = \theta_0$, one can compose a locally unbiased estimator of the statistical model $\{p_{\theta_0, \theta}(\cdot)\}_{\theta \in \Theta}$: Prepare a quantum state $\rho_{\theta_0}^n$, apply a sequence of quantum operations $\prod_{\kappa=1}^n \left\{ (\Lambda_{\theta_0, \theta_{x_\kappa}} \otimes \mathbf{I}) \circ \Xi_{\kappa, \theta_0}^n \right\}$, where $x_\kappa \sim p_{\theta_0, \theta}(\cdot)$ ($\kappa = 1, \dots, n$), measure the output state by $M_{\theta_0}^n$, and compute $T_{\theta_0, n}$. Therefore, due to classical estimation theory, the variance of $\mathcal{E}_{\theta_0, n}$ is $O(1/n)$. ■

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